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THE HISTORY OF ZENO'S ARGUMENTS ON MOTION:

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

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I.

A. THE PURPOSE OF ZENO'S ARGUMENTS.

Introduction. No questions on the foundations of mathematics are as old and of such perennial interest, reaching into the most recent speculations on the philosophy of mathematics, as are Zeno's arguments on motion. Zeno flourished in the fifth century before Christ, but only recently has G. Cantor's *Mengenlehre* been applied to the fuller elucidation of Zeno's paradoxes. The history of these paradoxes is largely the history of concepts of continuity, of the infinite and infinitesimal.

There has been great difference of opinion as to the exact nature and purpose of Zeno's arguments. None of Zeno's writings have come down to us. We know of his tenets only through his critics and commentators—Plato, Aristotle and Simplicius. Plato was born about 60 years after Zeno, Aristotle about 100 years after. Simplicius lived nearly 1,000 years after Zeno. Plato does not reproduce Zeno's arguments,[1] but discusses their purpose, which was "to protect the arguments of Parmenides against those who make fun of him and seek to show the many ridiculous and contradictory results which they suppose to follow from the affirmation of the *one*"; Zeno argues that "there is no *many,*" he "denies plurality."

Aristotle's Interpretation of Zeno's Arguments. Aristotle gives in very compressed form the arguments on motion, as they were handed down to him, in these words:[2]

[1] Plato, *Parmenides,* 127 D., Jowett's translation.

[2] Aristotle, *Physics,* VI, 9. See Carl Prantl's edition, in Greek and German, Leipzig, 1854. We omit the latter part of Prantl's version of Zeno's fourth argument against motion. The text is defective. Later we give Burnet's elaboration of the proof, which embodies the most probable conjecture of what Aristotle originally said.

"Zeno reasons here incorrectly; for, he says that everything, when in a uniform state, is continually either at rest or in motion, and that a body moving in space is continually in the *Now* [the instant], hence the arrow in its flight is at rest. But this is false, for the reason that time is not composed of individual, indivisible *Nows*, as also no other quantity is so composed. There are four proofs advanced by Zeno against motion, which present many difficulties to those who try to refute them. The first is the one on the impossibility of motion, on the ground that a thing moving in space must arrive at the mid-point before it reaches the end-point. We have gone into the details of this matter in our previous discussion. The second is the so-called Achilles; it consists in this that in a race the faster cannot overtake the slower; for, the pursuer must always first arrive at the point from which the one pursued has just departed, so that the slower is necessarily always a small distance ahead. But this is the same argument as that of bisection and differs from that merely in this, that the distance added is not divided quite into halves. That the slower is not overtaken follows from this argument, but it rests upon the same assumption as the bisection (for in both arguments it is stated that a thing cannot reach the end-point, since the quantity is divided in some manner. However, this second argument has the additional contention, that, in a race, even the most rapid cannot overtake the slowest and the refutation must therefore be the same. The claim that the one in the lead cannot be overtaken is false. To be sure, in the moment when he has the lead, he is not overtaken. Nevertheless he is overtaken; Zeno merely admits that the pursuer completely passes over the entire distance. These are two of his proofs; the third is the one referred to above, that the moving arrow is at rest. It is based on the assumption that time is made up of the individual *Nows*. If this is not admitted, then the conclusion does not follow. The fourth is in regard to equal bodies which move on a track parallel to other bodies of equal size but moving in opposite directions, namely the first moving thither from the end of the track, the second moving hither from the middle of it with the same speed. From this he thought that he must conclude that the half time must be equal to its double. The fallacy lies in the claim that when a body moves parallel to one in motion, with the same speed as it does move, passes one that is at rest, the time of passing is the same in both cases. This is false."

For greater clearness we repeat Zeno's arguments in the expanded form given by Burnet,[1] which is a free paraphrase of Aristotle's statements. We shall find it convenient, for future reference, to use the names "Dichotomy," "Achilles," "Arrow," and "Stade" for the four arguments against motion, respectively.

1. "Dichotomy": You cannot traverse an infinite number of points in a finite time. You must traverse the half of any given distance before you traverse the whole, and the half of that again before you can traverse the whole, and the half of that again before you can traverse it. This goes on *ad infinitum*, so that (*if space is made up of points*) there are an infinite number in any given space, and it cannot be traversed in a finite time.

2. "Achilles": The second argument is the famous puzzle of Achilles and the tortoise. Achilles must first reach the place from which the tortoise started. By that time the tortoise will have got on a little way. Achilles must then traverse that, and still the tortoise will be ahead. He is always nearer, but he never makes up to it.

3. "Arrow": The third argument against the possibility of motion *through a space made up of points* is that, on this hypothesis, an arrow in any given moment of its flight must be at rest in some particular point.[2]

[1] J. Burnet, *Early Greek Philosophy*, 1892, pp. 322 ff.

[2] A different version of the "arrow" is given by Diogenes Laertius, IX, 72, who lived 500 years after Aristotle, probably about 200 A.D.: "That which moves can neither move in the place where it is, nor yet in the place where it is not." In expanded form this argument is given by William Minto, professor of logic in the University of Aberdeen, in his *Logic, Inductive and Deductive*, London, 1893, p. 224, and by W. R. Royce Gibson in his *The Problem of Logic*, London, 1908, p. 290, as follows:

"If a body moves, it must move either where it is or where it is not." "But a body cannot move where it is; neither can it move where it is not."

"Therefore, it cannot move at all; i. e. motion is impossible."

4. "Stade": Suppose three parallel rows of points in juxtaposition, as in Fig. 1.

Fig. 1.	Fig. 2.
A · · · · · ·	←A · · · · · ·
B · · · · · ·	B · · · · · ·
C · · · · · ·	C · · · · · · →

One of these (B) is immovable, while A and C move in opposite directions with equal velocity so as to come into the position represented in Fig. 2. The movement of C relatively to A will be double its movement relatively to B, or, in other words, any given point in C has passed twice as many points in A as it has in B. It cannot, therefore, be the case that an instant of time corresponds to the passage from one point to another.

Tannery's Interpretation of Zeno's Arguments. As reported by Aristotle and Simplicius, Zeno's arguments are fallacies. That Zeno's reasoning was wrong has been the view universally held since the time of Aristotle down to the middle of the nineteenth century. During these many centuries the efforts of philosophers and mathematicians on this matter were to explain the exact nature of Zeno's blunders. More recently the opinion has been advanced that Zeno was incompletely and incorrectly reported, that his arguments were turned away from their true purpose by the Sophists who used them in advancing skepticism and the denial of knowledge, and that Aristotle described them as modified by the Sophists. Three great leaders in the interpretation of Greek thought, Cousin, Grote and P. Tannery, have construed Zeno's arguments as serious efforts, conducted with logical rigor. Cousin[1] maintained that Zeno successfully opposed the idea of *multiplicity devoid of all unity*. Grote[2] held a similar view. Tannery[3] argued that Zeno opposed the idea that *a point is unity in position*. Zeller[4] rejects all three explanations, mainly on the ground that they do not find support in the extant writings of Greek philosophers. It is also true that no Greek account definitely refutes any of the three interpretations. This lack of exact and detailed information is deplorable. Tannery's resuscitation of Zeno's arguments deserves our attention because of the internal coherence imparted to them. According to Tannery, Zeno did not deny motion, but wanted to show that motion was impossible under the conception of space as the sum of points. Tannery does not battle against the traditional statement that Zeno argued against plurality; he accepts Plato's general explanation, but differs from him and others on the precise nature of this plurality. According to Tannery it was not the ordinary notion that Zeno combated, according to which two lambs

This version of the "Arrow" must be rejected for two reasons: First, it occurs for the first time about 700 years after Zeno and is for that reason unreliable; second, there is no kernel to the argument. As Gibson says, it amounts to this: "If a body moves, it must move under conditions which render motion impossible."

[1] *Fragments philosophiques*, par M. Cousin. 5ème éd., Paris, 1865, p. 69.

[2] George Grote, *Plato*, Vol. I, 3d ed., London, 1875, pp. 100–104.

[3] Paul Tannery, "Le concept scientifique du continu. Zénon d'Elée et Georg Cantor," *Revue philosophique de la France et de L'Etranger*, X année, T. XX (1885), pp. 385–410; Paul Tannery, *Science hellène*, Paris, 1887, pp. 247–261.

[4] E. Zeller, *Die Philosophie der Griechen*, 1. Theil, 1. Hälfte, 5. Aufl., Leipzig, 1892, pp. 591–604.

are not one lamb, but a special notion of the Pythagoreans. Zeno's master, Parmenides, was attacked (says Tannery) by the Pythagoreans, and Zeno stepped into the conflict by battling against the mystical Pythagorean idea of a mathematical point—a point defined as *unity having position.* This Pythagorean definition is mentioned by Aristotle. Tannery interpreted this definition as signifying that a solid is the sum of points, just as a number is the sum of units. But such an idea is false. A point, mathematically speaking, is not unity or 1; it is a pure zero or 0. This interpretation of the phrase *unity having position* attributes to Zeno the grasp of an abstract concept, namely that of a point, destitute of length, breadth and thickness. That it is not unreasonable to ascribe to Zeno such abstractions seems evident from the following passage in Aristotle:[1]

"If the absolute unit is indivisible it would be, according to Zeno's axiom, nothing at all. For that which neither makes anything larger by its addition, nor makes anything smaller by its subtraction, is not one of the things that are, since it is clear that what *is* must be a magnitude, and, if a magnitude, corporeal, for the corporeal has being in all dimensions. Other things, such as the surface and the line, when added in one way make things larger, when added in another way do not; but the point and the unit do not make things larger however added."

It is hard to make out how much of this is the thought of Aristotle, and how much of it is Zeno's. Yet there would be no occasion to mention Zeno, had he no share in it.

It is believed by many critics that Zeno gave his arguments in the form of dialogue. Acting upon this view, Tannery entered upon the reconstruction of Zeno's arguments from the compressed passages handed down to us. Consider the following argument of Zeno on divisibility, as stated by Simplicius:[2]

"If that which is, has no magnitude it could not even be. Everything that truly is must needs have magnitude and thickness, and one part of it must be separated from another by a certain interval. And the same may be said of the next smaller part; it too will have magnitude, and a next smaller part. As well say this once for all as keep repeating it forever. For there will be no such part that could serve as a limit. And there will never be one part save in reference to another part. Thus, if the many have being, they must be both large and small—so small as to have no size at all, and so large as to be infinite."

Tannery's reconstruction of this passage is as follows:

A Pythagorean adversary claims that a finite quantity can be regarded as the sum of indivisible parts.

Zeno presents the first part of the dilemma resulting therefrom, thus: Admitting, as both of us do, that a quantity is infinitely divisible by continued bisection, it is evident that the parts become smaller and smaller. Hence, if there is a last term, it is 0. But the sum of such indivisible terms 0 is only 0. Hence the quantity has no magnitude.

But, says his adversary, why may the indivisible parts not be different from 0 and have magnitude?

[1] Aristotle, *Met.*, II, 4, 1001 b 7; translation taken from C. M. Bakewell, *Source Book in Ancient Philosophy*, New York, 1907, p. 23.

[2] Simpl. 140, 34 [R. P., 1050, Fr. 2 in Diels' arrangement]; C. M. Bakewell, *op. cit.*, p. 22.

Then Zeno presents the second part of the dilemma: If the indivisible parts have magnitude, and are infinite in number, the sum of these parts must be infinite.

Consequently, a finite quantity cannot be regarded as the sum of indivisible parts.

This explanation of Zeno's argument places Zeno certainly higher as a logician than does the old explanation which charged Zeno with inability to see that, if $xy = c$, x can increase and y simultaneously decrease in such a way that their product remains the same.

Now, let us see how Tannery applies *a point as unity in position* to the resuscitation of Zeno's arguments on motion. Tannery presents them in the form of a double dilemma.

The first argument, the "Dichotomy," involves matters which we have considered above, in connection with infinite division. As long as space is assumed to be made up of indivisible parts, the infinite number of parts, admitted by both contestants to result from continued bisection, cannot all be passed over in a given time.

The adversary may now present the point advanced by Aristotle, that the bisection is not carried on to actual infinity, but only to a potential infinity, and may therefore be run over in a finite time.

Zeno replies by stating the "Achilles," which does not involve bisection and in which the time-interval is subdivided in much the same way as the space-interval.

The adversary then takes the position that he has admitted too much. Finite time, he claims, is capable of division into an infinity of parts. Is there not a sum of instants? May there not correspond an instant to each successive position?

Against this Zeno directs his last two arguments, which constitute a second dilemma. At each instant the flying arrow occupies a fixed position. But occupying a fixed position at a given instant means that it is at rest that instant. Hence the arrow is at rest every instant of its flight.

The adversary explains that when saying that time was the sum of instants, he did not mean that each instant should apply to a fixed position of the arrow, but rather to the passage from each position to the next following position.

Here Zeno advances his "Stade" as his fourth argument. He shows that the demand of his adversary cannot be granted, because it would make all motions equal.

A motion from a point A (see Fig. 1 and Fig. 2) to the next point on the left requires one instant.

A motion from a point C to the next point on the right requires the same instant.

Hence A moves relatively to C twice as fast as relatively to B.

It is therefore not the passage from one point to the next that corresponds to the instant, for it would then follow that one is equal to its double.

Tannery's explanation of the four arguments, particularly of the "Arrow" and "Stade," raises these paradoxes from childish arguments to arguments with conclusions which follow with compelling force. It does not place Zeno in the position of being ignorant of the most simple ideas of relative motion; it exhibits Zeno as a logician of the first rank.

Tannery's conclusions have been strongly supported by G. Milhaud,[1] but opposed by other French writers and by Zeller.

CENTERS OF SIMILITUDE OF CIRCLES AND CERTAIN THEOREMS ATTRIBUTED TO MONGE. WERE THEY KNOWN TO THE GREEKS?

By RAYMOND CLARE ARCHIBALD, Brown University.

One of the most noticeable characteristics of French, German and Italian, as opposed to American, texts on elementary geometry is the emphasis laid on broad underlying principles. How many American high-school graduates could give one any idea of the theory of similitude of plane and solid figures? How many teachers realize the importance of this far reaching theory in the solution of geometrical problems, or are familiar with the equivalent of Petersen's excellent exposition?[2] At all events it seems well worth while to draw attention to some simple results in the theory, and to put on record their historical setting. The theorems attributed to Monge, which I propose to discuss, involve the centers of similitude of circles (spheres).

The centers of similitude of two circles (spheres), whose centers are A and B, are the points which divide AB internally, at C, and externally, at D, in the ratio of the radii, r_a, r_b ($r_a \geqq r_b$);

$$AC : CB = AD : BD = r_a : r_b.$$

The circles (spheres) may be situated in any fashion. If they are tangent (internally or externally), the point of tangency is a center of similitude. If concentric, we may, perhaps, say that either A, B, C, D coincide or else A, B, C coincide while D is indeterminate. If they are equal and non-concentric, D is at infinity[3] and C bisects AB.

Lines joining the ends of parallel radii pass through a center of similitude, and common tangent lines (planes), when they exist, also pass through such a center. Conversely, if through a center of similitude, D (or C), of two circles a line be

[1] G. Milhaud, "Le concept du nombre chez les Pythagoriciens et les Éléates," *Revue de métaphysique et de morale*, I, Paris, 1893, p. 141.

[2] J. Petersen, *Methods and Theories for the Solution of Problems of Geometrical Constructions*, Copenhagen, 1879, pp. 22 ff. This is an English edition of the remarkable Danish original. There are also French, German, Italian, Hungarian and Russian translations making in all some 15 editions. Because of its many notable qualities, this work stands preeminent in its special field.

[3] This is, of course, more an idea of *projective*, than of *elementary*, geometry.

drawn to cut the circles[1] in P, Q, P', Q', AP and BP', AQ and BQ' are pairs of parallel lines. There is similar obvious extension to spheres with tangent or secant lines.

In the first edition of his *Géométrie Descriptive*,[2] Monge derived the following results:

(A) *The six centers of similitude of three coplanar circles lie by threes on four straight lines.*

(B) *The vertices of the six common tangent cones of three spheres, taken in pairs, lie by threes on four straight lines.*

(C) *Given any four spheres in space fixed in magnitude and position, and the six cones tangent to them in pairs, externally; then the six vertices lie in a plane and indeed on four straight lines in the plane. If the six other tangent cones be drawn, then their vertices lie by threes in planes with threes of the first group.*

I believe that there has never been any question as to Monge's priority of discovery of theorem (C), about which I will add further comment presently. With regard to the particular case, theorem (A), (to which (B) is practically equivalent), a contemporary has been given some credit for its formulation. But, so far as I am aware, it has never before been suggested that it was known to the Greeks. In the following paragraphs I discuss these two views.

I.

Loria states:[3] "According to Fuss (*Nova Acta Petrop.*, T. XIV, 1805) Monge was inspired by d'Alembert." Kötter remarks:[4] "It has been noted by Grunert that Fuss published the Monge discussion [concerning circles], referring it back to d'Alembert," and a citation from *Nova Acta*, similar to the above, is given. Finally, as noticed by a correspondent of *L'Intermédiaire des Mathématiciens*,[5] according to Chasles,[6] Fuss attributes to d'Alembert the theorem for circles, while Carnot,[7] on the contrary, assigns it to Monge.

As Loria's statement is incorrect, while those of Grunert and Chasles are

[1] I consider the circle defined as a curve. It is to be noted, however, that this is *not* the definition generally used by modern writers. To be convinced of this it is sufficient to refer to the works of Enriques and Amaldi, Faifofer, Hadamard, and Ingrami.

[2] G. MONGE, *Géométrie descriptive. Leçons données aux écoles normales l'an de la republique* Paris . . . an VII [1798], pp. 54–55. In a new edition by Hachette, Paris, 1811, pp. 65–67. Hachette published in 1812, a supplementary volume, to pages 20–21 of which reference in this connection should be given. In English we have, *An elementary Treatise on descriptive geometry with a theory of shadows and of perspective: extracted from the French of G. Monge . . .* by J. F. Heather, London, 1851; our results are given on pp. 49–50. A free German translation of Monge's work, by G. Schreiber, was published at Karlsruhe and Freiburg, 1828. But a literal translation with notes, to which we will presently refer, was given in R. Haussner's edition, Ostwald's Klassiker, Nr. 117, Leipzig, 1900; our theorems occur on pages 68–70.

[3] In his history of "Perspektive und darstellende geomentrie," in CANTOR'S *Vorlesungen über Geschichte der Mathematik*, Bd. 4 (1908), p. 629.

[4] E. KÖTTER, *Die Entwickelung der synthetischen Geometrie*, Leipzig, 1901, p. 112.

[5] Question 1418, 1898, p. 271 and 1911, p. 29. No reply has so far been published.

[6] CHASLES, *Aperçu historique*, 3e éd., Paris, 1889, p. 293.

[7] CARNOT, *Mémoire sur la relation qui existe entre les distances relatives à cinq points quelconques pris dans l'espace*, suivi d'un *Essai sur la théorie des transversales*, Paris, 1806, p. 87.

only partially correct, it may be well to set forth exactly what Fuss did give in this connection.

The paper of Fuss, referred to above, was presented to the Academy, July 4, 1799, and entitled, "Demonstrations de quelques théorèmes de géométrie." It commences as follows:[1] "Several years ago a young Frenchman, then employed in the 'Corp *Imperial*' of the 'Cadets de Terre,' spoke to me about a geometrical theorem which, at the time when he was at Paris in the Royal military school, had some celebrity and was attributed to the late M. d'Alembert. Of this theorem I gave him a demonstration which I have recently found on looking over my papers." Fuss then takes up the theorem concerning the *external* centers of similitude of three circles (and this seems to be the theorem which he believed due to d'Alembert, although the more general one of Monge had been already published), before proceeding to properties "not less remarkable."

He considers four circles of different radii and in the same plane, and shows that the six *external* centers of similitude lie by threes on four straight lines. And, *for* n *coplanar circles, the* $n(n-1)/2!$ *external centers of similitude are situated, in general, by threes on* $n(n-1)(n-2)/3!$ *different straight lines.*

This is extended to n spheres with their centers in the same plane. The vertices of the cones tangent to the spheres in pairs (the centers of the spheres in each case on the same side of the vertex of the tangent cone) have exactly the properties indicated above for the centers of similitude.

Next, in taking up the corresponding theorems for a sphere, Fuss shows that: *If three small circles are enclosed in pairs by two great circles tangent to them the intersections of the three pairs of great circles are situated on the same great circle.* A similar theorem is given for n small circles on a sphere.

In none of the discussion are *internal* centers of similitude mentioned. On the other hand, as we have seen above, Monge stated three general theorems, namely (A), (B), (C), concerning both internal and external centers of similitude.

To make clear all that is implied in Theorem (C) it may be well to state the results symbolically.[2] If $E_{m,\,n}$ denote the external, and $I_{m,\,n}$ the internal centers of similitude of the spheres S_m and S_n ($m, n = 1, 2, 3, 4$ and $m \gtrless n$), the following groups of points lie in planes:

$$\left.\begin{array}{l} E_{1,\,2},\ E_{1,\,3},\ E_{1,\,4},\ E_{2,\,3},\ E_{2,\,4},\ E_{3,\,4}; \\ E_{1,\,2},\ E_{1,\,3},\ I_{1,\,4}\ \ E_{2,\,3},\ I_{2,\,4},\ I_{3,\,4}; \\ E_{1,\,2},\ I_{1,\,3},\ E_{1,\,4},\ I_{2,\,3},\ E_{2,\,4},\ I_{3,\,4}; \\ I_{1,\,2},\ E_{1,\,3},\ E_{1,\,4},\ I_{2,\,3},\ I_{2,\,4},\ E_{3,\,4}; \\ I_{1,\,2},\ I_{1,\,3},\ I_{1,\,4},\ E_{2,\,3},\ E_{2,\,4},\ E_{3,\,4}. \end{array}\right\}$$

Monge overlooked the three planes:[3]

[1] *Nova Acta Ac. Sc. imp. Petropolitanae*, Tome 14 (1797–1798), Petropoli, 1805, p. 139.

[2] *Cf.* HAUSSNER, *l. c.*, p. 200.

[3] LORIA makes two more errors (*l. c.*) by asserting that in theorem (C) "findet man einige Unrichtigkeiten, da Monge die Ebenen nicht betrachtete, von denen jede zwei innere und vier

$$\left.\begin{array}{l} I_{1,2},\ I_{1,3},\ E_{1,4},\ E_{2,3},\ I_{2,4},\ I_{3,4}; \\ I_{1,2},\ E_{1,3},\ I_{1,4},\ I_{2,3},\ E_{2,4},\ I_{3,4}; \\ E_{1,2},\ I_{1,3},\ I_{1,4},\ I_{2,3},\ I_{2,4},\ E_{3,4}. \end{array}\right\}$$

It is now apparent that so far as the testimony of Fuss[1] is concerned the name of d'Alembert should be associated with a very small portion, at most, of the theorem concerning the centers of similitude of three circles,[2] and that the terms "theorem of d'Alembert"[2] or "lines of d'Alembert,"[3] employed in connection with discussion of the four axes of similitude, are introduced both incorrectly and in a way to do injustice to discoveries of Monge.

II.

Let us now consider the question, Who was the discoverer of the centers of similitude[4] of two circles? Cantor asserts very definitely[5] that François Viète[6] [1540–1603] should be so considered. But from what follows it will be clear that many hundred years before Viète's time, Greeks were familiar with the points and several of their properties.

My argument is based almost wholly upon portions of the Mathematical Collections[7] of Pappus (*c.* 300 A. D.) and, more particularly, upon that part of it which describes the work *On Tangencies* by Apollonius of Perga (*c.* 225 B. C.). Among many other propositions which come up incidentally Pappus proves the following:

1. If two circles touch internally or externally and through the point of contact any two lines be drawn, the chords joining the points of intersection of these lines with each circle are parallel (pp. 832 f., 826 f.).

2. If two circles do not meet, the common direct tangent passes through the external center of similitude[8] (pp. 850 f.).

äussere Ähnlichkeitspunkte enthält." From the above it is clear that there are no "Unrichtigkeiten" in the theorem. Monge simply did not notice the three planes (here remarked) through four internal and two external centers of similitude (*not* two internal and four external as Loria asserts). In this connection Haussner also makes a slip (l. c., p. 199.)

[1] That is, in connection with the 1799 memoir cited above.

[2] I have vainly searched d'Alembert's *Opuscules mathématiques*, 8 tomes, Paris, 1761–1780, for any reference to this theorem.

[3] Used, for example, by F. G. M. *Exercices de Géométrie*, 5e éd., 1912, pp. 81, 146, 1260, 1272.

[4] The term *center of similitude* is due to Euler: "De centro similitudinis," *Nova Acta acad. sc. Petrop.*, Vol. 9 (1791), 1795, p. 154—"Conventuri exhib. die 23 Octob. 1777."

[5] M. Cantor, *Vorlesungen über Geschichte der Mathematik*, Bd. 2_2 (1900), pp. 590–591.

[6] F. Viète, *Apollonius Batavus*, Paris, 1600; reprinted in *Opera mathematica*, Lugd. Bat., 1646, pp. 325–338; also reprinted by J. W. Camerer in *Apollonii de tactionibus quae supersunt*, Gothae et Amstelodami, 1795, pp. 1–58 at end.

[7] Pappi Alexandrini Collectionis . . . edidit . . . Hultsch. Berolini, I (1876), pp. 195–201; II (1877), pp. 644–649; 820–853.

[8] Aristarchus (310–250 B. C.) gave a more general result. It is Proposition 1 of his work "On the Sizes and Distances of the Sun and the Moon": "Two equal spheres are comprehended by one and the same cylinder, and two unequal spheres by one and the same cone which has its vertex in the direction of the lesser sphere; and the straight lines drawn through the centers of the spheres is at right angles to each of the circles in which the surface of the cylinder, or of the cone,

3. If a circle C_3 is externally tangent to two other circles C_1, C_2, externally tangent to each other, the points of tangency of C_1C_3, C_2C_3, lie in a line with the external center of similitude of C_1C_2 (p. 208 f.).

4. The line joining the ends of two parallel radii drawn in opposite directions, of two equal circles, passes through the internal center of similitude (pp. 194 f.).

To sum up, we here find fundamental theorems concerning (*a*) internal and external centers of similitude of tangent circles; (*b*) the external center of similitude of unequal circles and (*c*) the internal center of similitude of equal circles. We have also, in 3, a particular case of part of Theorem (*A*) of Monge: *Given three circles C_1, C_2, and C_3, the internal centers of similitude of C_1C_3 and C_2C_3 are in a line with the external center of similitude of C_1 and C_2.*

I think further that it may be pretty conclusively shown, that the Greeks were familiar with the idea of the centers of similitude of two circles, in the general case and not alone in the particular cases referred to above. To make this clear I must recall a lemma given by Pappus in connection with his account of the second book of Apollonius *On Tangencies*. The main problem of this work is to describe a circle tangent to three given circles. If we regard lines and points as limiting cases of circles, we get ten types of problems, all of which were considered by Apollonius. In the second book only two types remain to be considered, namely:[1] (1) to describe a circle tangent to two given lines and to a given circle; (2) to describe a circle tangent to three given circles.

Now the famous lemma which Pappus gave for the solution of this latter problem is the following: "Given a circle in position, and the points D, E, F on a straight line; it is required to draw, to a point A of the circle, DA, AE to meet the circle again in B and C, such that BC lies in a line with CF."[2]

How can this lemma be used to solve the Problem of Apollonius? Pappus does not tell us. Viète and hundreds of others who have given solutions of the problem use methods in no wise involving the lemma; these methods can therefore bear little relation to that of Apollonius. I believe that Robert Simson, who did such signal service in the restoration of Euclid's *Porisms*, was the first to conjecture as to which one of the problems in *Tangencies* the lemma was subsidiary. He has written: "Often indeed have I revolved the subject in my mind, but I have never succeeded in arriving at any satisfactory conclusion; except that the lemma, by no uncertain marks, appeared to be necessary for the following problem: two circles and a point being given by position, it is required to describe a third circle which shall touch the given circles and pass through the given point.

touches the spheres." *Cf. Aristarchus of Samos the Ancient Copernicus*, ed. by T. L. Heath, Oxford, 1913, pp. 354–355. In the course of the proof of this proposition it is shown that the vertex of the cone is a center of similitude.

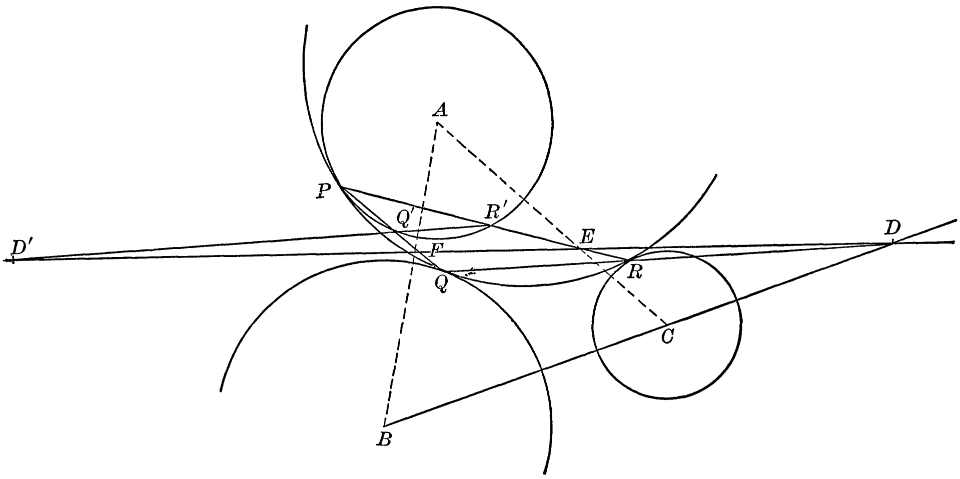
[1] Pappus, *l. c.*, pp. 646–7.

[2] The generalization of this lemma to the case where D, E, F, are not necessarily collinear, was first found by Robert Simson in 1731. The further generalization for any polygon inscribed in a circle was given by Giordano di Ottajano, a youth of sixteen, in 1784. Early in the nineteenth century, with the birth of the principle of duality and the development of projective geometry, the problem was solved for polygons circumscribed and inscribed to cones. Researches of Townsend, Potts and Renshaw led up to discoveries of Sir William Rowan Hamilton regarding polygons inscribed in a sphere, or an ellipsoid, or an hyperboloid, and with sides passing through given points.

In what manner, however, the lemma might be subsidiary to this problem I did by no means perceive. I have directed my attention to the solutions of Viète and others, hoping that by chance I might hit upon the analysis requiring this lemma; but in vain until this day, after various trials, I discovered the true analysis of Apollonius,—to which indeed both this Prop. 117 of Pappus as well as Props. 116 and 118 are manifestly subsidiary. February 9, 1734."[1] Simson then proceeds to set forth the analysis and synthesis of his solution after the Greek manner.

From editions of Pappus in his day, Simson could not learn that while the problem he attacked was in the first, the lemma he referred to was in the second, book of the *Tangencies*. With very similar reasoning, however, his method might have been extended to the case of a circle tangent to three given circles. Doubtless independent of Simson, this was first done, I believe, by Scorza in 1819.[2] In effect his method was the following.

Let A, B, C be the centers of the three given unequal circles which are also named by these letters. Let a, b, c be their respective radii. Moreover, suppose that the circles do not meet and that no one is inside another; then there are eight solutions corresponding to the different cases. It will suffice to consider a single case, when, say, the circle PQR is tangent to A internally at P, and to B, C externally, at Q, R respectively. Then QR passes through D the external center of similitude of B and C (Pappus, p. 210); and PR, PQ pass through E, F the internal centers of similitude of A and C, A and B respectively.



Let PR, PQ meet the circle A again in R' and Q', and let $R'Q'$ meet DF in D'. Then since QR is parallel to $Q'R'$ (Pappus, p. 832), D' is fixed by the ratio $DF/D'F = b/a$.

[1] R. Simson, *Opera Quaedem Reliqua*, Glasquae, MDCCLXXVI Appendix, pp. 20–23. *Cf.* T. S. Davies in *The Mathematician*, March, 1848, Vol. 3, p. 78.

[2] G. Scorza in "Divinazione della soluzione Apolloniana del problema de tre cerchi" by V. Flauti, *Atti della Reale Accademia delle Scienze e Belle Lettere*, Vol. 1 (1819), 77–78.

Hence, if E lies in the line DF,[1] the solution of the case of the problem we are considering is reduced to inscribing in the circle A a triangle $PQ'R'$, whose side produced shall pass through the fixed collinear points E, F, D'. This can be done by the lemma. The other cases may be treated in a similar way.

In the above I have neither paused to elaborate the details of the various steps nor endeavored to set forth the whole in the Greek manner. To fill in these lacunae, the interested reader may turn to the writings of Pappus, Simson, Flauti, Scorza, and Zeuthen[2]—for the weighty testimony of Zeuthen favors the solution just given as the most probable original of Apollonius.

In the course of this restoration not only have various properties of centers of similitude of circles, mentioned earlier in this paper, been used, but Monge's Theorem (A) has also been necessary. I therefore hold that Theorem (A) and centers of similitude were discovered by the Greeks. How much the more is the assertion concerning "Monge's Theorem" confirmed, when we recall that it follows immediately, on applying to the triangle ABC, the converse of the theorem of Menelaus of Alexandria (c. 80 A. D.) with regard to transversals.[3]

A CARDIOIDOGRAPH.

By C. M. HEBBERT, University of Illinois.

In his book called "Linkages"[4] J. D. C. DeRoos illustrates a rather complicated linkage for describing a cardioid. A simpler form of cardioidograph is presented in this paper.

The cardioid may be defined as the path traversed by any point of the circumference of a circle as it rolls upon a fixed coplanar circle of the same radius. In figure 1 consider the cardioid which is the path of the point B of the rolling

---

[1] Recalling the theorem of Aristarchus referred to above (note 8, page 9) a possible Greek mode of proof (followed by Monge) that D, E, F are collinear, is the following. Consider A, B, C as great circles of three spheres. If the two transverse planes tangent to the sphere A and to the spheres B and C, be drawn, these planes will each pass through the three centers of similitude, D, E, F; that is, these points lie on the line of intersection of the tangent planes. A second method of proof, in the Greek manner, is indicated below in note 3.

[2] H. G. ZEUTHEN, *Die Lehre von den Kegelschnitten im Altertum*, Deutsche Ausgabe besorgt von R. v. Fischer-Benson. Kopenhagen, 1886, pp. 381–383.

[3] In the above figure, for instance,

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=\frac{a}{b}\cdot-\frac{b}{c}\cdot\frac{c}{a}=-1,$$

and conversely. This is Lemma I, Book III, of the *Spherics* of Menelaus; or, to be more accurate, the lemma was stated in the form: If DEF is a transversal of the triangle ABC, the ratio of AF to FB is equal to the ratio compounded of the ratios CD to BD and CE to EA. (*Theodosii sphaericorum Lib. III . . . Menelai Sphaericorum Lib. III . . .* Messanae, 1558, pp. 36 *verso*—37 *recto* of second pagination, or p. 83 of *Menelai sphaericorum Libri III . . . curavit vir Cl. Ed. Halleius.* Oxonii, MDCCLVIII.)

[4] D. Van Nostrand, New York, 1879. English translation from the *Revue Universelle des Mines.*

circle. At the starting position shown by the dotted circle the point B was at A and the radius DB was in the position CA. Since arc AE = arc BE and the circles have equal radii, angle AOD = angle ODB. Also, $DB = \frac{1}{2}OD$. These two equalities are fundamental in the construction of the linkage.

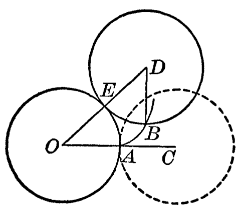


FIG. 1.

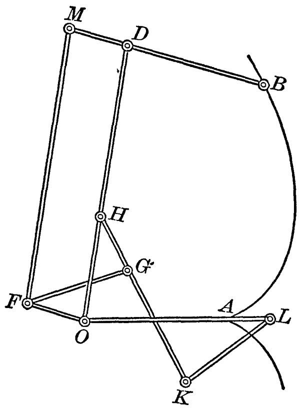


FIG. 2.

The links, BM, MF, DO, FG, HK, KL, OL, OF, are pinned together at D, M, F, G, H, K, L, and O, so that $OD = 2DB$; $DM = OF = HG$; $FM = OD$; $FG = OH = KL$; $HK = OL$; and $OH : HK = HG : OH$ (Fig. 2).

From the similar triangles OHK, OLK, OHG, and OFG, we have angle HOL = angle HOF.[1] In the parallelogram $OFMD$, angle FOD = angle ODB, i. e., angle AOD = angle ODB.

Hence, if the part OL is kept fixed the linkage compels the point B to move according to the two equalities derived from the definition of the cardioid, and a pencil fastened at B will describe a cardioid if OL is kept fixed while the other parts are free to move.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY, University of Minnesota.

One Thousand Exercises in Plane and Spherical Trigonometry. By EDWIN S. CRAWLEY. Published by E. S. CRAWLEY, University of Pennsylvania, Philadelphia, Pa., 1914, v+70 pages. $0.50.

Professor Crawley, who is the author of a text-book on trigonometry, has recently published this book of exercises to be used with any text-book. The exercises are intended for class-room drill, and are therefore for the most part of an average grade of difficulty. The first 950 exercises are arranged in a classified list as follows: The measurement of angles (Ex. 1–25); The functions of one

[1] On the theory of linkages in general consult Arnold Emch: "Kinematische Gelenksysteme und die durch sie erzeugten geometrischen Transformationen," Solothurn, 1907.

angle (Ex. 26–125); Plane right triangles (Ex. 126–150); Problems involving the solution of plane right triangles (Ex. 151–305); Identities involving more than one angle. Inverse functions, etc. (Ex. 306–525); Plane oblique triangles (Ex. 526–550); Problems involving the solution of plane oblique triangles (Ex. 551–625); Problems involving areas, altitudes, etc., of triangles, and the radii of circles related to a triangle (Ex. 626–700); Trigonometric equations (Ex. 701–800); Spherical right triangles, and quadrantal triangles, and applications (Ex. 801–870); Formulas of spherical triangles (Ex. 871–900); Spherical oblique triangles and applications (Ex. 901–950).

Exercises 950–1,000 are of a miscellaneous character. Answers to all the exercises are given at the end of the book.

W. H. Bussey.

The Theory of Numbers. By R. D. Carmichael. John Wiley & Sons, New York, 1914. 94 pages. $1.00.

This little volume (No. 13 of the "Mathematical Monographs" edited by M. Merriman and R. L. Woodward) contains in the first seventy-five pages an excellent introduction to the classical multiplicative theory of numbers, while an additional chapter of seventeen pages gives a brief account of the meaning of some advanced topics, such as Fermat's Last Theorem, Theory of Quadratic Residues and Galois' Imaginaries, all of which are too technical for a detailed treatment in a short introductory book.

The volume would seem to the reviewer to be particularly valuable for self-study. Any reader who will thoroughly master it will have gained a clear idea of the fundamental questions of the theory, and will be well prepared for the study of more complete or more advanced treatises.

A valuable feature is the list of 84 problems which are distributed among the various chapters. Many of these are of interest in themselves, and their careful study will do much to make the abstract theory gain a concrete meaning.

The titles of the chapters will serve to indicate the subjects treated:

(1) Elementary properties of integers. (2) On the indicator of an integer. (3) Elementary properties of congruences. (4) The theorems of Fermat and Wilson. (5) Primitive roots modulo m. (6) Other topics.

The last chapter is the only one where references are given. Although the first five chapters are quite self-contained, an occasional reference would be useful, even to readers familiar with the subject. For example, the proof of Wilson's theorem, as given on p. 50, while it may be well known, is new to the reviewer. On pages 50–51 it might well be stated in connection with Wilson's theorem (although commonly omitted in books on number-theory) that the trivial congruences:

$$(n-1)! \not\equiv 0 \text{ mod. } n, \text{ for } n \text{ prime},$$

$$(n-1)! \equiv 0 \text{ mod. } n, \text{ for } n \text{ not prime}, n > 4,$$

furnish also a complete test for prime numbers.

The following misprints and slips were noticed:
p. 14, last formula: read $(mnp + n\alpha + m\beta)$ for $(mnp + \alpha + \beta)$;
p. 22, last line: read $0 \leqq b_i < n$ for $0 < b_i < n$;
p. 28, exercise 8: read " is " for " as ";
p. 93, Index: read " Primitive ϕ-roots " for " Primitive ψ-roots."
In p. 29, line 2, the word " simplest " may be a little misleading.

The author is to be congratulated on his happy selection of material which permits him, in this small volume, to give a clear insight into the methods employed in elementary number-theory, as well as to present in a well-connected fashion the principal results.

A. J. KEMPNER.

UNIVERSITY OF ILLINOIS.

Trigonometry. By MAXIME BÔCHER and HARRY DAVIS GAYLORD. New York, Henry Holt and Company, 1914. ix + 142 pp.

The authors have stated in the preface that "this text-book has been prepared with a view to giving an adequate treatment of what is essential, in a form sufficiently concise so that the real simplicity and brevity of the subject may be in evidence," and that "the number of principles involved is very limited and should not be artificially multiplied by the formulation of rules." This ought to be the aim of every text-book on elementary trigonometry, but mathematicians will differ in their opinions as to what constitutes real simplicity and brevity. In reviewing this text its object must be kept in mind. We must also not overlook the fact, implied in the preface, that the text is not intended primarily for college classes, but also for beginners in the subject.

Chapter I, covering 19 pages, after defining the six trigonometric functions for acute angles, etc., treats among others the following topics: the use of trigonometric tables, solution of right triangles, projections of line segments, line values and the variation of trigonometric functions for angles of the first quadrant, simple trigonometric identities and the solution of easy trigonometric equations.

Chapter II introduces the student to logarithms. The logarithm of a number is first defined for the base 10, and the four fundamental principles are derived for logarithms of this base. In a subsequent section, in small type, the logarithm of a number is defined for any base, and the formula for the change of base is given, the method of derivation being indicated by a numerical example. Throughout the text the logarithms of numbers less than unity are written with negative characteristics and positive mantissas, the usual method being mentioned on page 25. The co-logarithm is used extensively. A student who uses the rule for the logarithm of a quotient and does not use co-logarithms, will find himself confronted with an expression of the type, $\bar{4}.42357 - \bar{3}.96284$, which the text does not adequately consider.

Chapter III devotes 24 pages to the consideration of the topics: directed angles of any magnitude, directed line segments, the trigonometric functions of angles of any magnitude, etc., closing with a few examples in trigonometric identities, the solution of trigonometric equations and a brief note on inverse

trigonometric functions. The addition of the paragraph entitled, "Alternative Form of Definition," page 36, is commendable and will aid the student in understanding the line values of the tangent and secant of the second quadrant angle, Fig. 23, page 38. The omission of the laws for the addition of directed line segments and the addition of directed angles is to be regretted. Section 22 treats of the important principles whereby the functions of any angle can be expressed in terms of the functions of an angle not greater than 45°. Only two pages are given to this work. One figure, illustrating the method of proof for the case $A + 90°$, A being taken acute, is all the student has to aid him in grasping the contents of the section. The principles are nowhere formulated. The chief criticism against this chapter is that it leaves the bulk of the work to be done by the instructor and makes the student dependent upon notes taken in class, instead of giving him a clearly written, well illustrated chapter on the subject. Students, in general, are not investigators, and many examples well explained and illustrated, whenever possible, are necessary to impress firmly the principles upon their minds.

The next chapter deals with the solution of oblique triangles. The last two chapters are devoted to the study of spherical geometry, the formulæ of spherical trigonometry and the solution of spherical triangles. At the end of the text are to be found numerous and excellent exercises which are arranged to correspond with the chapters and their sections.

The text has an individuality and a newness of presentation which are attractive and stimulating, yet it is the opinion of the reviewer that real simplicity and clearness have been sacrificed for brevity.

J. C. Rayworth.

Washington University,
St. Louis, Mo.

Algebraic Invariants. Mathematical Monographs edited by Mansfield Merriman and Robert S. Woodward. No. 14. By Leonard Eugene Dickson, Professor of Mathematics in the University of Chicago. John Wiley & Sons, New York, 1914. x + 100 pages. $1.00.

The series of monographs to which the present volume belongs is one of the most praiseworthy undertakings relating to American mathematical publications. It is very lamentable that some American mathematicians have been discouraged from preparing treatises on higher mathematics by the fact that American publishers have often hesitated to assume the expense of publication unless the particular work under consideration would promise to be a financial success. This has made us more dependent upon works in foreign languages than would otherwise have been the case, and it has had a bad indirect influence. In fact, at least two very eminent and extensive mathematical works which were prepared in this country have been published in a foreign language.

To the credit of American publishers it should however be stated that in recent years there have appeared in America a number of treatises of such high mathematical standing as to make it seem unlikely that these particular works

could by themselves be financial successes. It is hoped that the day is coming when some enterprising American mathematical publishers will realize that leadership implies sacrifices, such as our big retail merchants have been making in their efforts to increase patronage. There is a great need for large mathematical treatises, encyclopedias, and other works of reference in the English language. Our mathematical development is to-day impeded more by the lack of such works than by any other one obstacle.

Monographs like the one under review are filling a great want in our literature, as they tend to create an interest in some of the rich fields of modern mathematics. Their brevity tends to encourage some who might be overawed by the extent of a comprehensive treatise. Professor Dickson seems to have been very successful in the choice of the material for the present monograph. Beginning with very simple examples from plane analytic geometry, the author furnishes a number of geometrical interpretations and applications of invariants and covariants in the first part of the book, which covers 29 pages.

Part II is devoted to the theory of invariants in non-symbolic notation and "treats of the algebraic properties of invariants and covariants, chiefly of binary forms; homogeneity, weight, annihilators, seminvariant leaders of covariants, law of reciprocity, fundamental systems, properties as functions of the roots, and production by means of differential operators." In Part III the symbolic notation of Aronhold and Clebsch is explained and illustrated by simple examples. Great care has been taken to present the matters in a clear manner and to avoid the difficulties which the beginner frequently encounters in this field.

A considerable number of exercises and illustrative examples are provided throughout the book, and there is a good index at the end of the volume. The author's deep mathematical insight combined with his special efforts to present matters in a clear manner have resulted in a monograph on this important subject which the beginners in this field, as well as those who have already made considerable progress therein, should read with unusual interest and profit.

G. A. Miller.

SOLUTIONS OF PROBLEMS.

Edited by B. F. Finkel and R. P. Baker.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

415. Proposed by C. N. SCHMALL, New York City.

Show that

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}.$$

I. Solution by Hermon L. Slobin, University of Minnesota.

Let us expand $f(x) = x$ as a Fourier cosine series,

$$f(x) = a_0/2 + a_1 \cos x + a_2 \cos 2x + \cdots.$$

Hence, we have

$$a_m = \frac{2}{\pi}\int_0^\pi x \cos mx dx = \frac{2}{m^2\pi}[(-1)^m - 1] \qquad m \neq 0,$$

and

$$a_0 = \frac{2}{\pi}\int_0^\pi x dx = \pi.$$

Hence,

$$x = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right].$$

Putting $x = 0$, we have

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots.$$

Note.—See also Byerly's *Fourier Series and Spherical Harmonics*, p. 40, where this special case is handled by means of a Fourier sine series. EDITORS.

II. SOLUTION BY THE PROPOSER.

Let

$$\Sigma_1 = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots;$$

and

$$\Sigma_2 = \frac{1}{1^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \cdots.$$

Then

$$\Sigma_1 = \frac{1}{1^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \cdots + \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \cdots$$

$$= \frac{1}{1^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \cdots + \frac{1}{2^n}\left(\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots\right)$$

or,

$$\Sigma_1 = \Sigma_2 + \frac{1}{2^n}\Sigma_1, \qquad \therefore \Sigma_2 = \frac{2^n - 1}{2^n}\Sigma_1,$$

Whence, Σ_2 is known when Σ_1 is found. When $n = 2$, Σ_1 is known to be $\pi^2/6$; then

$$\Sigma_2 = \frac{3}{4}\frac{\pi^2}{6} = \frac{\pi^2}{8}.$$

Note.—The value $\Sigma_1 = \pi^2/6$ is deduced in Loney's *Trigonometry*, page 444, where also the value $\Sigma_2 = \pi^2/8$ is deduced by means of the development of $\cos\theta$ as an infinite product. Such a solution was given by A. M. HARDING, A. L. MCCARTY and the PROPOSER. Loney also uses the same process to find $1/1^4 + 1/3^4 + 1/5^4 + \cdots = \pi^4/96$ and $1/1^4 + 1/2^4 + 1/3^4 + \cdots = \pi^4/90$. EDITORS.

III. SOLUTION BY W. C. BRENKE, University of Nebraska.

Consider the series,

$$S(x) \equiv \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots,$$

which reduces to the given series when $x = 0$.

This series may be summed. Put

$$S_n(x) \equiv \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \cdots + \frac{\cos (2n + 1)x}{(2n + 1)^2}.$$

Differentiating twice with respect to x, we have

$$S_n'(x) = -\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \cdots + \frac{\sin (2n + 1)x}{2n + 1}\right],$$

and

$$S_n''(x) = -[\cos x + \cos 3x + \cdots + \cos (2n + 1)x].$$

Multiplying both members of the last equation by $2 \sin x$ and expanding the products on the right by use of the formula, $2 \cos mx \sin x = \sin (m + 1)x - \sin (m - 1)x$, we have

$$2 \sin x S_n''(x) = -\sin (2n + 2)x.$$

Hence,

$$S_n'(x) = -\int \frac{\sin (2n + 2)x}{2 \sin x} dx + c_1.$$

Letting $n = \infty$, the integral vanishes as is easily seen on integrating by parts, and we have

$$S'(x) = c_1, \qquad 0 < x < \pi.$$

Hence

$$S(x) = c_1 x + c_2, \qquad 0 < x < \pi.$$

To determine the constants, note that $S'(\pi/2) = -(\pi/4)$, as follows at once from the series for $\tan^{-1} x$ when $x = 1$. Hence $c_1 = -(\pi/4)$. Also, since $S(\pi/2) = 0$ we get $c_2 = \pi^2/8$. Therefore,

$$S(x) = -\frac{\pi}{4}x + \frac{\pi^2}{8},$$

or

$$\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots = -\frac{\pi}{4}x + \frac{\pi^2}{8}, \qquad 0 < x < \pi.$$

Since the series converges uniformly for all values of x, we may put $x = 0$, which gives the required result.

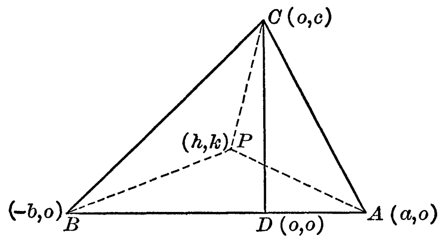
GEOMETRY.

441. Proposed by H. E. TREFETHEN, Colby College.

In the triangle ABC find the locus of points at which the sides AB and AC subtend equal angles.

I. Solution by C. N. Schmall, New York City.

Take the base AB and the altitude CD as the axes of coördinates. Then we have to find the locus of a point P such that $\angle APB = \angle APC$. Let the coördinates be A: $(a, 0)$, B: $(-b, 0)$, C: $(0, c)$. The slope of the line joining (x_1, y_1) and (x_2, y_2) is $(y_2 - y_1)/(x_2 - x_1)$. Hence the slope m_1 of PB is $k/(h + b)$; the slope m_2 of PA is $k/(h - a)$; and the slope m_3 of PC is $(k - c)/h$. Therefore, $\angle APB = \tan^{-1}[(m_1 - m_2)/(1 + m_1m_2)]$, which, on substituting the values of m_1 and m_2, becomes



$$\tan^{-1}\frac{k(h - a) - k(h + b)}{(h - a)(h + b) + k^2}. \tag{1}$$

Likewise,

$$\angle APC = \tan^{-1}\frac{m_2 - m_3}{1 + m_2m_3} = \tan^{-1}\frac{hk - (k - c)(h - a)}{h(h - a) + k(k - c)}. \tag{2}$$

Hence from (1) and (2), writing x, y for h, k, we have the required locus of P:

$$\frac{y(x - a) - y(x + b)}{(x - a)(x + b) + y^2} = \frac{xy - (y - c)(x - a)}{x(x - a) + y(y - c)}.$$

Upon further reduction, this becomes

$$(2a + b)y^3 + c(x - b - 2a)y^2 + (2ax + bx + ab)y + c(x - a)^2(x + b) = 0,$$

an equation of the third degree.

Also solved by V. M. Spunar.

II. Solution by Nathan Altshiller, University of Washington, Seattle.

The problem may be generalized: Find the locus of the point M such that the two given segments AB, $A'B'$ subtend at M equal positive angles (MA, MB), (MA', MB').

If the fixed positive angle (MA, MB) turns about the point M, its sides MA, MB generate two projective pencils, whose double elements are the isotropic lines MC, MC' (C, C' designate the cyclical points) passing through M. From

this projectivity results the involution $M(AA', BB', CC')$. Hence the problem takes the following projective form: Given three couples of points AA', BB', CC', find the locus of the point M such that the three couples of rays $M(AA', BB', CC')$, shall be in involution. Cayley proved analytically[1] that the locus of M is a general cubic passing through the given points. The following is an outline of a synthetic proof of this theorem and provides a method for the construction of the locus.

The four points A, A', B, B' determine a complete quadrangle (Q_1) whose diagonal triangle is PQR $[P \equiv AA', BB';\ Q \equiv AB', A'B;\ R \equiv AB, A'B']$. Let (Q_2) be the complete quadrangle in which C, C' are two vertices and Q, R two diagonal points, S denoting the third diagonal point. The two quadrangles (Q_1), (Q_2) determine two pencils of conics (π) and (σ), which determine the same involution (I) on the line QR, the points Q and R being the double elements of (I). Let π_n and σ_n be those conics of the two pencils which determine the same couple N, N' of (I). Besides N, N' the two conics have two other points M, M' in common. M and M' belong to the required locus. The line MM' passes through the fixed point L which is the point of intersection of the two rays that correspond to the ray QR in the two involutions $Q(AA', CC')$ and $R(AA', CC')$. The point $T_n \equiv (MM', QR)$ is the harmonic conjugate of the fixed point $T \equiv (PS, QR)$ with respect to the couple NN'. The pencil of rays (L) and the pencil of conics (σ) are thus projective and the locus of the points of intersection M, M' of homologous elements of these two forms is a cubic (Chasles' theorem).

The points of the given couples may be real or conjugate imaginaries.

If the points of one of the couples, say A and A', coincide, this point will be a double point on the cubic. Such is the case in the proposed question. The required locus is therefore a circular cubic (C_3) with a double point at A (a strophoid) passing through the vertices B and C of the given triangle.

The cubic (C_3) is the pedal curve of the point A with regard to the parabola (P), which is tangent: (1) to the line BC; (2) to the bisectors C, C' of the angle BAC and its adjacent angle; (3) to the perpendiculars n, n' erected at B and C to AB and AC respectively.

Proof.—Let O, O' be the centers of two arcs of circles at all points of which the two segments AB, AC subtend equal angles. Besides A the two circles meet in a point M which belongs to (C_3). P and P' being the points diametrically opposite A in the two circles, the line PP', according to a known property of intersecting circles, passes through M and is perpendicular to AM. The points O, O' being on the perpendicular bisectors of the segments AB, AC, the points P, P' are on the perpendiculars n, n'. The angles BAO and CAO' are equal; therefore the lines AP and AP' form a couple of conjugate elements in the equilateral involution of rays at A, the double elements of which are the bisectors C, C'. Hence

$$(P \cdots) \;\overline{\wedge}\; A(P \cdots) \;\wedge\; A(P' \cdots) \;\overline{\wedge}\; (P' \cdots).$$

[1] A. Cayley, *Collected Papers*, I, p. 184.

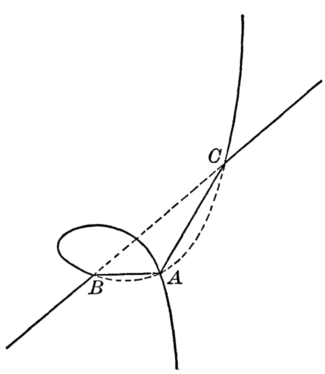
The tangents to (C_3) at A are the bisectors C, C'. The foot of the perpendicular from A upon BC belongs to (C_3).

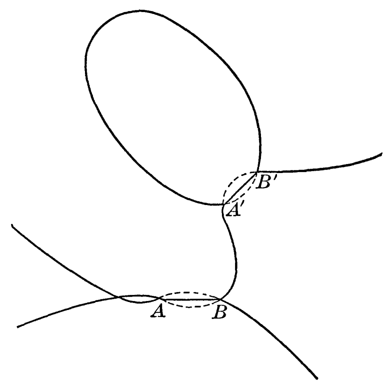
Remarks, I. In order that all the points of (C_3) shall belong to the required locus the condition of " equal angles " must be replaced by " equal positive angles." Otherwise only those parts of (C_3) that are included in the angle BAC and in its vertical angle, will posses the required property.

II. If $AC = AB$, the cubic (C_3) degenerates into the bisector of the angle BAC and the circumference of the circle circumscribed about the triangle ABC.

III. The line of centers OO' envelops a parabola similar to the parabola (P), the point A being the center of similitude.

Editorial Note.—The formulas of analytic geometry and the procedure of synthetic geometry are both such that continuous deformations do not destroy their validity. The word " subtend," having physical connotations, is not so flexible.





The locus in general, for the usual meaning of subtend, is made up of parts of two cubic curves forming a continuous curve in the projective plane without a continuous derivative. In the special case one cubic degenerates and the real part is the line BC. The heavy curves in the accompanying cuts sufficiently show the locus.

443. Proposed by C. N. SCHMALL, New York City.

A quadrilateral of any shape whatever is divided by a transversal into two quadrilaterals. The diagonals of the original figure and those of the two resulting (smaller) figures are then drawn. Show that their three points of intersection are collinear.

Solution by Laenas G. Weld, Pullman, Ills.

[The word "(smaller)" should be omitted, as it destroys the generality of the proposition.]

Using trilinear coördinates, let ABC be the triangle of reference and designate the points in which the lines $l\alpha + m\beta + n\gamma = 0$ and $p\alpha + q\beta + r\gamma = 0$ cut the sides a, b, c by L, M, N and P, Q, R, respectively. Then $BCMN$, $BCQR$ and $QMNR$ are the three quadrilaterals in question. The equations of the diagonals are readily written as follows:

$$BM: \quad l\alpha + n\gamma = 0, \qquad BQ: \quad p\alpha + r\gamma = 0,$$

$$CN: \quad l\alpha + m\beta = 0, \qquad CR: \quad p\alpha + q\beta = 0,$$

$$RM: \quad lp\alpha + lq\beta + np\gamma = 0,$$

$$QN: \quad lp\alpha + mp\beta + lr\gamma = 0.$$

At the intersections of BM and CN, of BQ and CR, and of RM and QN, the coördinate ratios are respectively:

$$\alpha_1 : \beta_1 : \gamma_1 :: \quad -mn \quad : \quad nl \quad : \quad lm,$$

$$\alpha_2 : \beta_2 : \gamma_2 :: \quad -qr \quad : \quad rp \quad : \quad pq,$$

$$\alpha_3 : \beta_3 : \gamma_3 :: l^2rq - p^2mn : p^2nl - l^2rp : p^2lm - l^2pq.$$

The vanishing of the determinant of these three sets of coördinates proves the proposition.

Also solved by NATHAN ALTSHILLER, H. C. FEEMSTER, and F. M. MORGAN.

444. Proposed by S. A. COREY, Hiteman, Iowa.

Let $ABCDE$ be a pentagon, plane or gauche, with sides AB, BC, CD, DE, and EA. Bisect BC and DE in H and K respectively. Extend AB from B to B', and AE from E to E'. On AB' take sects AP and AV, and on AE' take sects AL and AT. Draw AD, AC, AH, AK, and DT. Let a, b, c, and d equal AL/AE, AT/AE, AV/AB, and AP/AB, respectively. Extend (or contract) AC from C to W, and AD from D to S, making $AW = a \times AC$ and $AS = d \times AD$. Draw LM and PN parallel to, and of the same currency as, AD and AC respectively, and of lengths $c \times AD$ and $b \times AC$, respectively. Draw AM, AN, ST, and WV. Draw DQ and VX parallel to, and of the same currency as, CB and TS, respectively. We are to prove that $2(ad + bc)(AK \times AH \times \cos KAH + KE \times HC \times \cos QDK) = AM \times AN \times \cos MAN + TS \times VW \times \cos WVX$.

SOLUTION BY THE PROPOSER.

Let KE, HB, AH and AK, in the proposed figure, be represented by the vectors x, y, z, and w, respectively. Then by vector addition, $w + x = AE$, $w - x = AD$, $z + y = AB$, $z - y = AC$; by construction, $\angle ST \cdot WV = \angle WVX$, $\angle KE \cdot HB = \angle QDK$, $(w + x)a = AL$, $(w - x)c = LM$, $(z + y)d = AP$, $(z - y)b = PN$, $(w + x)b = AT$, $(w - x)d = AS$, $(z + y)z = AV$, and $(z - y)a = AW$; also by vector addition, $(w + x)a + (w - x)c = AM$, $(z + y)d + (z - y)b = AN$, $(w + x)b - (w - x)d = ST$, and $(z + y)c - (z - y)a = WV$.

Consider now the algebraic identity,

$$\begin{aligned} &[(w + x)a + (w - x)c][(z + y)d + (z - y)b] \\ &\quad + [(w + x)b - (w - x)d][(z + y)c - (z - y)a] = 2(ad + bc)(wz + xy) \end{aligned} \tag{1}$$

and note that when fully expanded each term is of the second degree in x, y, z, and w. Also note that the identity may be written

$$AM \times AN + ST \times WV = 2(ad + bc)(AK \times AH + KE \times HB) \tag{2}$$

if we substitute vectors as above indicated.

Inasmuch as vector multiplication is commutative if no term of the product is of a degree higher than the second in the vectors employed, and if the scalar part only of the resulting product be considered, we may assume that the algebraic identity (1) has a geometric interpretation which may be derived from (2) by considering the scalar part only of the vector products indicated in (2). The scalar part of the product of two vectors may be taken as the positive product of the lengths of the vectors into the cosine of their included angle. Placing this interpretation on the scalar part of the products indicated in (2) the equation of the problem is at once obtained, and the truth of the theorem established.

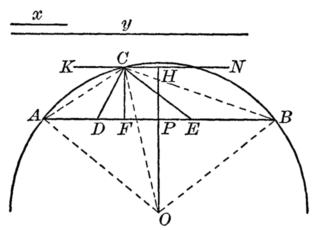
As no assumption has been made restricting any of the lines to any one plane the pentagon may, of course, be either plane or gauche. To help form a mental picture of a gauche pentagon, let $ABCD$ be a tetrahedron, with edges AB, BC, CD, DA, AC, and BD, and let E be a point within or without such tetrahedonr. Then if E be connected with A and D by right lines the figure $ABCDE$ will be a gauche pentagon.

445. Proposed by CLIFFORD N. MILLS, South Dakota State College.

Given the perimeter of a right triangle and the perpendicular falling from the right angle on the hypotenuse, to determine the sides of the triangle.

SOLUTION BY G. I. HOPKINS, Manchester (N. H.) High School.

Let x be the altitude and y the perimeter. Draw $AB = y$, and OH its $\perp$ bisector. Make $PH = x$, and draw KN through $H \perp$ to HO. Make $PO = AP$. With O as center and radius OA describe the circle ACB. Draw the chords CA and CB, and the radii OA, OB, and OC. Make $\angle ACD = \angle CAD$, and $\angle BCE = \angle CBE$. $\therefore DCE$ is the $\triangle$ required. For,



$$\angle AOP = 45° = \angle POB. \quad \therefore \angle AOB = 90°.$$

$\angle OAC = \angle OCA$ and $\angle CAD = \angle ACD$. $\therefore \angle OCD = \angle OAD = 45°$. In like manner $\angle OCE = 45°$; $\therefore \angle DCE = 90°$, $AD = DC$, and $EB = CE$; $\therefore$ the perimeter of the $\triangle DCE = y$, CF is $\perp$ to AB, $\therefore CF = HP = x$.

Note. The figure is not accurate as OP is not made equal to AP.

Also solved by B. J. BROWN, A. H. HOLMES, C. N. SCHMALL, and NATHAN ALTSHILLER.

CALCULUS.

356. Proposed by F. B. FINKEL, Drury College.

A steel girder l ft. long and w ft. wide is moved along a passageway a ft. wide and into a corridor at right angles to the passageway. How wide must the corridor be to admit the girder?

The equation of the line RS parallel to PQ and distant ω from it is:

$$y = \frac{dy_i}{dx_i}x + a^{1/3}l^{2/3} + \omega\sqrt{1 + \left(\frac{dy_i}{dx_i}\right)^2}.$$

Substituting and simplifying,

$$y = -\frac{a^{1/3}x}{[l^{2/3} - a^{2/3}]^{1/2}} + a^{1/3}l^{2/3} + \frac{wl^{1/3}}{[l^{2/3} - a^{2/3}]^{1/2}}.$$

The intersection of this line with the line $y = a$ is at $x = (l^{2/3} - a^{2/3})^{3/2} + w(l^{1/3}/a^{1/3})$, which is the required width.

Also solved by W. A. FLANAGAN and L. SIVIAN.

357. Proposed by W. D. CAIRNS, Oberlin College.

A continuous variable represented by a point on a vertical line changes according to such a law that it is reduced to $1/m$ of its value on being moved a units upward, irrespective of the special position from which it is moved. Find the law of change, that is, the relation between the variable y and the height h of the variable point above a fixed point of the vertical line.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

We are to solve the functional equation $f(h + a) = (1/m)f(h)$. Taking logarithms of both sides we have the equation, $\log f(h + a) = \log f(h) - \log m$. Let $\psi(h) = \log f(h) + (h/a)\log m$. Then $\log f(h + a) = \psi(h + a) - [(h + a/a]\log m$, and $\log f(h) = \psi(h) - (h/a)\log(m)$. The above logarithmic equation becomes $\psi(h + a) = \psi(h)$, which is satisfied by any periodic function of period a. Then the function $f(h) = e^{\psi(h)-(h/a)\log m} = m^{-(h/a)} \cdot e^{\psi(h)}$ or $f(h) = m^{-(h/a)} \cdot P(h)$ where $P(h)$ is any periodic function of period a.

Also solved by JOSEPH NYBERG, J. W. CLAWSON, and the PROPOSER.

MECHANICS.

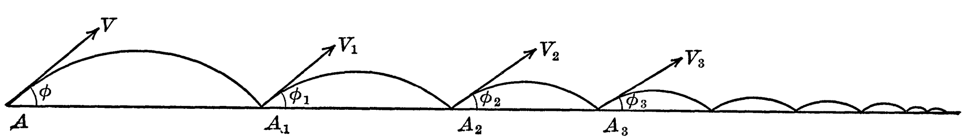
289. Proposed by C. N. SCHMALL, New York City.

A particle of elasticity e is projected with a velocity v at an angle of elevation φ from a point on a smooth horizontal plane. Show that after $\dfrac{2v\sin\varphi}{g(1 - e)}$ seconds it will cease to rebound and will move along the plane with a uniform velocity $v\cos\varphi$.

SOLUTION BY A. M. HARDING, University of Arkansas.

Let t_1 = time from A to A_1, t_2 = time from A_1 to A_2, etc. Equating the normal components before and after each impact we obtain

$$\begin{gathered} v_1\sin\varphi_1 = ev\sin\varphi, \qquad v_2\sin\varphi_2 = ev_1\sin\varphi_1 = e^2v\sin\varphi, \\ v_3\sin\varphi_3 = ev_2\sin\varphi_2 = e^3v\sin\varphi, \text{ etc.} \end{gathered} \tag{1}$$



Hence, $t_1 = (2v \sin \varphi)/g$, $t_2 = (2v_1 \sin \varphi_1)/g = (2ev \sin \varphi)/g$, $t_3 = (2v_2 \sin \varphi_2)/g = (2e^2 v \sin \varphi)/g$, etc.

$$T = t_1 + t_2 + t_3 + \cdots = \frac{2v \sin \varphi}{g}[1 + e + e^2 + e^3 + \cdots] = \frac{2v \sin \varphi}{g(1-e)}.$$

Equating tangential components before and after each impact we obtain

$$\begin{gathered} v_1 \cos \varphi_1 = v \cos \varphi, \qquad v_2 \cos \varphi_2 = v_1 \cos \varphi_1 = v \cos \varphi, \\ v_3 \cos \varphi_3 = v_2 \cos \varphi_2 = v \cos \varphi, \text{ etc.} \end{gathered} \tag{2}$$

From (1) and (2) we have

$$v_1^2 \sin 2\varphi_1 = ev^2 \sin 2\varphi, \qquad v_2^2 \sin 2\varphi_2 = e^2 v^2 \sin 2\varphi, \text{ etc.}$$

$$\begin{aligned} \text{Hence Total Range} &= R_1 + R_2 + R_3 + \cdots \\ &= \frac{v^2 \sin 2\varphi}{g} + \frac{v_1^2 \sin 2\varphi_1}{g} + \frac{v_2^2 \sin 2\varphi_2}{g} + \cdots \\ &= \frac{v^2 \sin 2\varphi}{g}[1 + e + e^2 + \cdots] = \frac{v^2 \sin 2\varphi}{g(1-e)}. \end{aligned}$$

Since the plane is smooth the particle will move with a uniform velocity $v \cos \varphi$ after it ceases to rebound.

Also solved by ELIJAH SWIFT, L. SIVIAN, CLIFFORD N. MILLS, HORACE OLSON, A. H. WILSON, and J. W. CLAWSON.

NUMBER THEORY.

195. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

Find triangles whose sides are integers and one of whose angles is 60°.

II. SOLUTION OF THE GENERALIZED PROBLEM BY R. A. JOHNSON, Western Reserve University.

It is evident that there is no solution, unless the cosine of the given angle is rational. If, however, this is the case, there are always solutions. Let us assume $\cos \theta = m/n$ where m and n are relatively prime integers, such that $n > 0$, $|m| < n$.

Now if a, b, c be sides of a triangle in which θ is the angle opposite a, we have

$$b^2 - 2\frac{m}{n}bc + c^2 = a^2. \tag{1}$$

Let

$$a = b - \frac{q}{p}c \tag{2}$$

where, as is always possible, p and q have no common factor. Moreover, it is obviously sufficient to consider only *positive* values of p and q; for a case in which they would have unlike signs may be reduced to this case by a change of notation,

i. e., interchanging b and c. Making the substitution and solving, we obtain

$$\frac{c}{b} = \frac{2p(mp - nq)}{n(p^2 - q^2)}. \tag{3}$$

Combining (2) and (3),

$$\frac{a}{b} = \frac{n(p^2 + q^2) - 2mpq}{n(p^2 - q^2)}. \tag{4}$$

It does not follow, however, that when (4) is true c/b necessarily has the form (3). We find by substituting from (4) in (1) that

$$\frac{c}{b} = \frac{2p(mp - nq)}{n(p^2 - q^2)} \quad \text{or} \quad \frac{2q(np - mq)}{n(p^2 - q^2)} \tag{5}$$

and each of these leads to the value (4) for a/b. Hence these expressions yield the only possible solutions, and we have as necessary conditions for solutions: Either

$$(6_1)\ \begin{cases} a_1 = \dfrac{X}{Y_1}[n(p^2 + q^2) - 2mpq], \\[2ex] b_1 = \dfrac{X}{Y_1}[n(p^2 - q^2)], \\[2ex] c\ \ = \dfrac{X}{Y_1}[2p(mp - nq)], \end{cases} \quad \text{or} \quad (6_2)\ \begin{cases} a_2 = \dfrac{X}{Y_2}[n(p^2 + q^2) - 2mpq], \\[2ex] b_2 = \dfrac{X}{Y_2}[n(p^2 - q^2)], \\[2ex] c_2 = \dfrac{X}{Y_2}[2q(np - mq)], \end{cases}$$

where Y_i $(i = 1, 2)$ is the highest common factor of the corresponding set of three expressions in brackets, and X is any positive integer. It remains to determine what limitations must be imposed on p and q in order that these formulas shall actually yield solutions.

The numbers a_1, b_1, c_1 will all be positive, if and only if

$$m > 0, \qquad \frac{q}{p} < \frac{m}{n},$$

and a_2, b_2, c_2 are all positive if and only if

$$q < p.$$

When these conditions are satisfied, we easily show that in either case we have

$$a + b > c, \quad \text{and} \quad b - a < c$$

so that a, b, c are sides of a triangle. Moreover,

$$\cos A \equiv \frac{b^2 + c^2 - a^2}{2bc} = \frac{m}{n},$$

and we actually have a solution.

It will be noted that the expressions in brackets in formulas 6_1 and 6_2 will often have a common factor. In the well-known special case of a right triangle, we need not compute in the case where a common factor exists, for it is known that by a different choice of p and q we get a similar triangle whose sides are prime numbers. Mr. Martin, in the solution above referred to, assumes (apparently correctly) that this is true also when $m/n = \frac{1}{2}$. But in the general case this conclusion is erroneous. That is, if we use the formulas

$$(7) \qquad \begin{aligned} \bar{a} &= n(p^2 - q^2) - 2mpq, & \bar{b} &= n(p^2 - q^2), \\ \bar{c}_1 &= 2p(mp - nq), & \bar{c}_2 &= 2q(np - mq), \end{aligned}$$

we see that

(a) in general, neither of the sets (a, b, c) has a common factor that could be removed from the expressions;

(b) there exist triangles which cannot be expressed in the form (7) by any choice of p and q subject to the limitations we have determined.

For, adding, we have

$$\bar{a} + \bar{b} + \bar{c}_1 = 2p(p - q)(m + n),$$

$$\bar{a} + \bar{b} + \bar{c}_2 = 2(p + q)(np - mq),$$

and, for example, if $a + b + c$ be a prime number no choice of p and q will satisfy either one of these alternative necessary conditions. It follows that there are solutions of our problem, sets having no common factor, that must be obtained by deriving from formulas 6_1 and 6_2 sets having common factors, then dividing out such factors.

We summarize as follows:

Let m and n be any two integers without common factor, $n > 0$, $|m| < n$. If a, b, c be three integers without a common factor, representing lengths of the sides of a triangle in which $\cos(b, c) = m/n$, then the following formulas always give sets (a, b, c), and they give all possible sets:

I: (applicable only when $m > 0$)

$$\left\{\begin{aligned} a &= \frac{1}{F}[n(p^2 + q^2) - 2mpq], \\ b &= \frac{1}{F}[n(p^2 - q^2)], \\ c &= \frac{1}{F}[2p(mp - nq)], \end{aligned}\right. \qquad \left\{\begin{aligned} a &= \frac{1}{F}[n(p^2 + q^2) - 2mpq], \\ b &= \frac{1}{F}[n(p^2 - q^2)], \\ c &= \frac{1}{F}[2q(np - mq)], \end{aligned}\right.$$

where p, q are any two relatively prime positive integers, such that in I $q/p < m/n$, in II $q < p$ and in each case F is the highest common factor of the three brackets.

II: (applicable for all values of m)

We can tabulate all possible common factors of either set of three brackets, namely; the following, and no others, will be common factors: in several cases, 2 is a common factor; any factor common to $(m + n)$ and $(p + q)$; any factor common to $(m - n)$ and $(p - q)$; in I, any factor common to n and p; in **II**, any factor common to n and q.

An excellent solution of the special problem was received from S. A. JOFFE.

210. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

If a and b are relatively prime and $(a + b)$ is even, then

$$(a - b)ab(a + b) \equiv 0 \pmod{24}.$$

SOLUTION BY L. C. MATHEWSON, Urbana, Ill.

Since a and b are relatively prime, at most only one can be even; furthermore, since their sum is even, they must both be odd. Accordingly, let

$$a = 2a_1 + 1, \qquad b = 2b_1 + 1.$$

Then

$$\begin{aligned}(a - b)(a + b) &= 4a_1^2 - 4b_1^2 + 4a_1 - 4b_1\\ &= 4(a_1 + b_1 + 1)(a_1 - b_1).\end{aligned}$$

If, now, a_1 and b_1 are both even or both odd, $a_1 - b_1$ is divisible by 2 and thus $(a - b)(a + b)$ divisible by 8. If either a_1 or b_1 is odd (the other even), then $a_1 + b_1 + 1$ is divisible by 2 and thus $(a - b)(a + b)$ divisible by 8. Hence, in either case, under the conditions of the problem

$$(a - b)ab(a + b) \text{ is divisible by } 8.$$

Should at least one of the two given numbers be divisible by 3, then the solution would be evident. If neither a nor b is divisible by 3, then

$$a \equiv 1 \text{ or } 2, \text{ mod } 3, \qquad \text{and} \qquad b \equiv 1 \text{ or } 2, \text{ mod } 3.$$

If both a and b are congruent to 1 or to 2 (mod 3), then $(a - b)$ is divisible by 3; if one is congruent to 1 and the other to 2, then $(a + b)$ is divisible by 3. Hence, in all cases under the conditions of the problem

$$(a - b)ab(a + b) \text{ is divisible by } 3.$$

Therefore, since $(a - b)ab(a + b)$ is divisible by 8 and also by 3,

$$(a - b)ab(a + b) \equiv 0, \pmod{24}.$$

Also solved by S. W. REAVES, H. T. BIGELOW, S. A. JOFFE, ELIJAH SWIFT, HORACE OLSON, E. E. WHITFORD, and H. C. FEEMSTER.

QUESTIONS AND DISCUSSIONS.

EDITED BY U. G. MITCHELL, University of Kansas.

This department was established as a medium for the discussion of questions relating to the teaching of collegiate mathematics and to incidental difficulties encountered by investigators. The publication of such questions and of replies to them naturally stimulates interest on the part of others whose work may be touched by these or by similar questions and difficulties.

Interest in mathematics, however, is largely stimulated by the publication of papers and a reader often feels that he could add something of value by way of extension or brief comment, but fears that what he has to offer may seem scarcely worthy of presentation as a formal paper. If, at such a time, he felt that there was some suitable place for publishing his contribution, he would probably prepare and submit it. It might, indeed, happen that a considerable part of such contributions would not prove suitable for publication; but it might also happen that many of the articles would not only furnish much of value to others but also frequently lead to larger and more extended investigations on the part of the author.

It has been suggested that this department ought to be broad enough to cover such discussions, and, believing the suggestion to be a good one, we are, accordingly, changing the heading to " Questions and Discussions " and publishing below two discussions which are due to interest aroused by the publication of "A Geometrical Discussion of the Regular Inscribed Heptagon " by J. Q. McNatt, in the January, 1914, issue.

In assuming charge of the department the new editor wishes to thank the retiring editor, Professor Carmichael, for kindly assistance, and to express the hope that all friends of the department will continue their generous coöperation and support.

All correspondence relating to this department should hereafter be addressed to U. G. MITCHELL, 1313 Massachusetts St., Lawrence, Kansas.

DISCUSSIONS.

Relating to the inscribed Heptagon, Nonagon and Undecagon.

(Cf. Article by J. Q. McNatt, this MONTHLY, Vol. XXI, pp. 13–14.)

I. DISCUSSION BY CLIFFORD N. MILLS, Brookings, S. D.

Let ADE be an equilateral triangle inscribed in a circle of unit radius and suppose the arc subtended by the side AD to be divided into three equal parts at points B and C. Draw the diameter HE and the chords AB, BC and CD. Also let fall a perpendicular BM from B to side AD. Let $2a$ be the length of the side of the regular inscribed nonagon.

To compute $2a$ in terms of the radius as a unit.

From the figure it is readily seen that

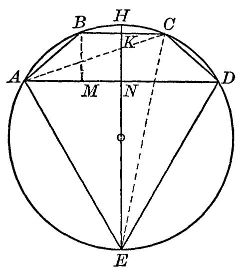
$$NE = 3/2, \qquad HN = 1/2, \quad \text{and} \quad \overline{BK}^2 = (3/2 + KN)(1/2 - KN).$$

But since $BK = a$ and $KN = \sqrt{3a^2 + a\sqrt{3} - 3/4}$

$$a^2 = 3/4 - \sqrt{3a^2 + a\sqrt{3} - 3/4} - 3a^2 - a\sqrt{3} + 3/4.$$

Simplifying gives

$$16a^4 + 8a^3\sqrt{3} - 12a^2 - 4a\sqrt{3} + 3 = 0.$$



Solving by Horner's method

$$a = .34202 \cdots,$$

which gives $2a = .68404\ldots$ as the side of the regular nonagon inscribed in a circle of radius unity.

Approximate Construction of a Nonagon.

By means of similar triangles it can be proved that $CD = CE - AC$ and that CD is approximately $2/5$ of AD. Hence, if the side of an inscribed equilateral triangle be divided into five equal parts, a radius equal to the length of two of these parts is the approximate length of chord for inscribing a regular nonagon. In fact, the $2/5$ of the side of the inscribed equilateral triangle is slightly less than $77/76$ of the side of the inscribed regular nonagon.

II. Discussion by J. A. Colson, Searsport, Maine.

The "Heptagon Cubic" as given by Mr. J. Q. McNatt, in the January number of this Monthly, can be readily found by trigonometry from the following data:

$$h = 2 \sin \pi/7, \qquad \text{and} \qquad \sin \theta = \sin (\pi - \theta).$$

Put $\theta = 3\pi/7$, then $\pi - \theta = 4\pi/7$, and $\sin 3\pi/7 = \sin 4\pi/7$, or $3 \sin \pi/7 - 4 \sin^3 \pi/7 = (4 \sin \pi/7 - 8 \sin^3 \pi/7)\sqrt{(1 - \sin^2 \pi/7)}$.

Since $\sin \pi/7 \neq 0$, we may divide the above equation by $\sin \pi/7$, thus obtaining

$$3 - 4 \sin^2 \pi/7 = (4 - 8 \sin^2 \pi/7)\sqrt{(1 - \sin^2 \pi/7)}.$$

Square both members of this equation, unite similar terms, and we have

$$7 - 56 \sin^2 \pi/7 + 112 \sin^4 \pi/7 - 64 \sin^6 \pi/7 = 0.$$

But $\sin \pi/7 = h/2$. Hence by substitution, we obtain the "Heptagon Cubic"

$$7 - 14h^2 + 7h^4 - h^6 = 0.$$

This same equation can also be obtained by taking $\theta = 2\pi/7$ and $\pi - \theta = 5\pi/7$ or by taking $\theta = \pi/7$ and $\pi - \theta = 6\pi/7$ and proceeding in a manner similar to the above.

We may find the *Nonagon Cubic* by the same process. For, let n be a side of a regular nonagon inscribed in a circle whose radius is unity. Then $n = 2 \sin \pi/9$. Also $\sin \pi/3 = \sin (\pi - \pi/3) = \sin 2\pi/3 = 2 \sin \pi/3 \cos \pi/3$. Hence, dividing throughout by $\sin \pi/3$, we have

$$1 = 2 \cos \pi/3 = 2(1 - 4 \sin^2 \pi/9)\sqrt{(1 - \sin^2 \pi/9)}.$$

Square both members of this equation, unite similar terms, and we have

$$3 - 36 \sin^2 \pi/9 + 96 \sin^4 \pi/9 - 64 \sin^6 \pi/9 = 0.$$

Substitute for $\sin \pi/9$ its value $n/2$, and we have for the *Nonagon Cubic*

$$3 - 9n^2 + 6n^4 - n^6 = 0.$$

Hence, by Horner's Method, we have

$$n^2 = .467911113762043929595215,$$

and finally

$$n = .684040286651337466088199.$$

The *Undecagon Quintic* can be found in the same way. For let u be a side of a regular undecagon inscribed in a circle whose radius is unity. Then $u = 2 \sin \pi/11$.

Also $\sin 5\pi/11 = \sin (\pi - 5\pi/11) = \sin 6\pi/11$. Proceeding as before we find for the *Undecagon Quintic*

$$11 - 55u^2 + 77u^4 - 44u^6 + 11u^8 - u^{10} = 0.$$

Hence, by Horner's Method, we have

$$u^2 = .3174929343376376622763767,$$

and finally

$$u = .5634651136828593954228358.$$

Replies to Questions.

17. In analytic geometry, simplicity and directness are gained by making the condition for the collinearity of three points and the equation of the straight line depend upon the determinant formula for the area of a triangle. Similar advantages are gained by making the condition of complanarity of four points and the equation of the plane depend upon the determinant formula for the volume of a tetrahedron. The former is given in the texts. Why should not the latter be given? A uniform method of developing these two determinants is desired from some contributor.

Reply by A. M. Kenyon, Purdue University.

Let ABC be a triangle whose vertices are (x_1, y_1), (x_2, y_2), and (x_3, y_3) and let A', B', C', be the projections on the x-axis of these points. We may suppose that the notation has been so chosen that $x_1 \leq x_2 \leq x_3$; and we assume at first that the vertices of the triangle all lie in the upper half plane.

$$\text{Area } ABC =^{1} \pm [\text{Area } A'ACC' - \text{Area } A'ABB' - \text{Area } B'BCC']$$
$$= \pm \tfrac{1}{2}[(y_1 + y_3)(x_3 - x_1) - (y_1 + y_2)(x_2 - x_1) - (y_2 + y_3)(x_3 - x_2)]$$

or, setting $y_1 + y_2 + y_3 = s$,

$$= \pm\frac{1}{2}\left[-(s - y_2)\begin{vmatrix} x_1 & 1 \\ x_3 & 1 \end{vmatrix} + (s - y_3)\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} + (s - y_1)\begin{vmatrix} x_2 & 1 \\ x_3 & 1 \end{vmatrix}\right]$$

$$= \pm\frac{1}{2}\begin{vmatrix} s - y_1 & x_1 & 1 \\ s - y_2 & x_2 & 1 \\ s - y_3 & x_3 & 1 \end{vmatrix} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

If the triangle does not lie wholly in the upper half plane, we may choose a new x-axis, at a distance a below the old one such that this condition is satisfied; then,

$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 + a & 1 \\ x_2 & y_2 + a & 1 \\ x_3 & y_3 + a & 1 \end{vmatrix} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

The symmetry of the result shows that it is independent of the order of choice of the vertices except as to the sign factor. The + sign holds when A, B, C, is the positive direction of the perimeter of the triangle; i. e., such that if one traversed the perimeter passing through the vertices in the order A, B, C, the area inside the triangle would lie always on the left.

Let $ABCD$ be a tetrahedron whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4); and let A', B', C', D', be their projections on the xy-plane. We may suppose the notation to have been so chosen that A', B', C', is the positive direction of the perimeter of the triangle $A'B'C'$, and that D' lies within the angle $A'B'C'$. We assume that the vertices of the tetrahedron all lie on the positive side of the xy-plane. Then,

$$\text{Vol. } ABCD =[2] \pm [\text{Vol. } A'B'C'ABC - \text{Vol. } A'B'D'ABD - \text{Vol. } B'C'D'BCD [3] \pm \text{Vol. } C'A'D'CAD].$$

By means of the rule for finding the volume of a truncated triangular prism, this reduces, on setting $z_1 + z_2 + z_3 + z_4 = s$, as follows:

$$\text{Vol. } A'B'C'ABC = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\frac{s - z_4}{6}, \quad \text{Vol. } A'B'D'ABD = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}\frac{s - z_3}{6}.$$

$$\text{Vol. } B'C'D'BCD = \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}\frac{s - z_1}{6}, \quad \text{Vol. } C'A'D'CAD = \pm\begin{vmatrix} x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}\frac{S - z_2}{6};$$

[1] The + sign holds when B and B' are on the same side of AC, and the − sign when they are on opposite sides.

[2] The + sign holds when D and D' are on the same side of the plane ABC, and the − sign when they are on opposite sides.

[3] The + sign holds when D' is outside the triangle $A'B'C'$, and the − sign when it is inside.

where in the last, the $+$ or $-$ sign is to be taken according as D' is inside or outside the triangle $A'B'C'$. Substituting these values above,

$$\text{Vol. } ABCD = \pm\frac{1}{6}\begin{vmatrix} s - z_1 & x_1 & y_1 & 1 \\ s - z_2 & x_2 & y_2 & 1 \\ s - z_3 & x_3 & y_3 & 1 \\ s - z_4 & x_4 & y_4 & 1 \end{vmatrix} = \pm\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

If the tetrahedron is not wholly above the xy-plane, we may choose a new xy-plane below the old one such that this condition is satisfied and, as in the case of the triangle, show that the formula still holds.

The result is again independent of the order of choice of the vertices except as to sign which is affected by a change of order in an obvious manner.

14. In the process of solving a certain physical problem Professor H. S. Uhler, of Yale University, was led to the definite integral

$$\int_0^a (a^2 - x^2)x dx \int_{a-x}^{a+x} \frac{e^{-cy}}{y} dy,$$

for which he found the value

$$\frac{1}{c^2}\left[a^2 - \frac{3a}{c} + \frac{1}{c}\left(a + \frac{3}{2c}\right)(1 - e^{-2ac})\right],$$

a and c being positive constants. Professor Uhler would like to see how other persons attack the problem of evaluating this integral.

(Note: Several good solutions have been received in reply to the above request one of which was published in December and others may be published later; but since there are some who would be interested to see Professor Uhler's method we give it below.—Editor.)

REPLY BY H. S. UHLER, Yale University.

Let

$$I \equiv \int_0^a (a^2 - x^2)x dx \int_{a-x}^{a+x} \frac{e^{-cy}}{y} dy.$$

Differentiate I with respect to the parameter c, then

$$\frac{dI}{dc} = -\int_0^a (a^2 - x^2)x dx \int_{a-x}^{a+x} e^{-cy} dy,$$

which can be evaluated at once and gives

$$\frac{dI}{dc} = \frac{2a^2}{c^3} - \frac{6a}{c^4} + \frac{6}{c^5} - \left(\frac{6a}{c^4} + \frac{6}{c^5} + \frac{2a^2}{c^3}\right)e^{-2ac}.$$

We can now integrate both sides of this equation, thus

$$\int_0^I dI = \int_c^\infty \left\{\frac{2a^2}{c^3} - \frac{6a}{c^4} + \frac{6}{c^5} - \left(\frac{6a}{c^4} + \frac{6}{c^5} + \frac{2a^2}{c^3}\right)e^{-2ac}\right\} dc$$

whence

$$I = \frac{1}{c^2}\left[a^2 - \frac{3a}{c} + \frac{1}{c}\left(a + \frac{3}{2c}\right)(1 - e^{-2ac})\right].$$

NOTES AND NEWS.

Edited by W. D. Cairns.

Dr. W. A. Hurwitz has been promoted to an assistant professorship of mathematics at Cornell University.

Dr. Daniel Buchanan is assistant professor of mathematics at Queen's University, Kingston, Ontario.

Dr. H. E. Buchanan is professor of mathematics at the University of Tennessee.

The Michigan State Teachers' Association met at Kalamazoo, October 29, 30, 31. Mr. L. P. Jocelyn, of the Ann Arbor High School, presented a paper before the High School section devoted to the terminology of mathematics. Professor L. C. Karpinski, of the University of Michigan, presented a paper before the same section on correlation between arithmetic, algebra and geometry.

A table of *Natural Sines* to every second of arc, and to eight places of decimals, computed by E. Gifford, was recently printed by Abel Heywood & Son, Manchester, England. The table covers 540 pages and is based on the well-known tables of Rheticus which were computed during the sixteenth century and have been revised by many later writers.

Washington University, St. Louis, includes in its courses in descriptive geometry a treatment of the subject of axonometry. Does any of our readers know other institutions where work in this line is given?

In the number of the *Bibliotheca Mathematica*, dated August 6, 1914, G. Eneström makes additional remarks on modifications and corrections relating to the well-known *Vorlesungen über Geschichte der Mathematik* by Moritz Cantor. On page 281 of this number it is stated that unless one verifies the statements in Cantor's history one should regard them with suspicion. The corrections which have been noted in the *Bibliotheca Mathematica* have already reached the surprisingly large number of about 2,000.

H. E. Timerding has recently issued a pamphlet entitled "Die Verbreitung mathematischen Wissens und mathematischer Auffassung" as part of the large series of books published by B. G. Teubner, Leipzig, Germany, under the general title "Die Kultur der Gegenwart." This pamphlet has been reviewed critically by G. Eneström in a recent number of the *Bibliotheca Mathematica.*

The students of mathematics at the University of Illinois maintain two mathematical clubs. One of these is mainly for undergraduates and first year graduates, and holds its meetings once a month. The other meets about twice a month to consider reports and papers by members of the faculty and advanced graduate students. For the past year Mr. C. M. Hebbert was chairman of the former, while Professor G. A. Miller was chairman of the latter of these two clubs.

So few works relating directly to the history of mathematics have been published in Italian that such a work as *Archimedes e il suo tempo*, by P. Midolo (Syracuse, Prem. Tipografia del "Tamburo," xxv+523 pp., 1912), is especially noteworthy. As the title indicates the author is concerned chiefly with the life and times of Archimedes. In the preface explicit statement is made that the book, dedicated to the cultured citizens of Syracuse, is not a work of erudition, but for the populace and the young student. However, the writer adds that neither is this a work of fantasy. A very readable account is presented, not only of the life but also of the works of Archimedes. Some effort is made to give a fairly complete bibliography, but prominent among the omissions are the articles by Tannery and Heath's *Archimedes*. The history of Syracuse is treated extensively, and also the general topic of mathematical instruction in ancient Greece.

In *School Review* for October Mr. E. R. Breslich, of the University of Chicago High School, makes answer to the reported criticism of Superintendent Francis of the Los Angeles schools: "God bless the girl who refuses to study algebra. It is a study that has caused many a girl to lose her soul." Mr. Breslich maintains that the friends of algebra have in recent years brought about notable changes, in that much abstract work is omitted or postponed to a later stage; that the subject is being vitalized by bringing it into touch with real life; that the processes of algebra are now illustrated and represented concretely by using space material. He thinks that pupils, under these circumstances, will have a fair chance to find out whether they like the subject and are able to do the work successfully, and that it may prove wise, on account of the abstract character of algebra, to teach algebra and geometry side by side. He concludes that those who protest against the old should help us to formulate the new.

During the autumn and winter quarters a course in advanced calculus is offered by Professor E. J. Wilczynski in the University College of the University of Chicago. This course is given at a time which will suit the convenience of high school teachers of mathematics in Chicago and vicinity.

The *Columbia University Quarterly* for September 1914 contains a tribute to the American astronomer George William Hill, who lectured for several years on celestial mechanics at Columbia University. A good portrait accompanies the article.

A paper by Professor G. A. Miller entitled "Recent mathematical activities" appears in the November number of *Popular Science Monthly*. In this article Professor Miller comments, among other matters, on Sundman's solution of the three-body problem, the various American mathematical journals, the mathematical encyclopedias, and the present year's activity in the International Commission on the Teaching of Mathematics.

In Volume 10 of the eleventh edition of the *Encyclopedia Britannica*, under the word Fermat, there appears the following statement: "He died in the belief that he had found a relation which every prime number must satisfy, namely

$2^{2^n} + 1 =$ a prime." It is very evident that most of the small prime numbers are not of this form. In the noted French encyclopedia, called *Nouveau Larousse*, there appears, under the word substitution, the following statement: "The notion of group due to Galois has opened a vast field." Every one ought to know that the notion of group is much older than the work of Galois. As these statements appear in works which are generally regarded as very reliable they are of especial interest. (G. A. Miller, in *School Science and Mathematics*.)

In memory of Professor J. L. GILPATRICK, professor of mathematics in Denison University from 1873 until his death in 1912, the alumni society has raised a scholarship fund of $1,200.

Professor O. E. GLENN has been promoted to a professorship in mathematics in the University of Pennsylvania and Dr. H. H. MITCHELL to an assistant professorship, while Dr. L. J. REED has been appointed instructor in mathematics.

According to *Science*, the University of Louvain has accepted the offer of Cambridge University to give the free use of its libraries, laboratories and lecture rooms during the present crisis, in order that the work of the Belgian University as a corporate body may be carried on without breach of continuity. Cambridge University has only 1,500 students, as against 3,500 last year, and other institutions report a similar shrinkage. The German universities have about one third of their usual attendance.

Miss MARIAN E. DANIELLS, formerly a graduate student at the University of Chicago, and later teacher of mathematics in the Fort Wayne, Indiana, high school, has accepted an instructorship in mathematics at Iowa State College, Cedar Falls.

After ten years of successful experience, the Mathematical Club of Syracuse University has been reorganized into a mathematical fraternity, Pi Mu Epsilon, whose aims are the advancement of mathematics and scholarship. The fraternity was incorporated under the laws of the state of New York under date of May 25, 1914. The charter members consist of members of the mathematical faculty, graduate students in mathematics, and undergraduate major and minor mathematical students. Among the powers granted under the articles of incorporation is that of granting charters to other chapters to be organized elsewhere.

At Washington University, Mr. Raymond DuHadway has been appointed to an instructorship in mathematics for the year 1914–15, to fill the vacancy made by the resignation of Mr. James E. Donahue.

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THE HISTORY OF ZENO'S ARGUMENTS ON MOTION:

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

II.

By FLORIAN CAJORI, Colorado College.

B. ARISTOTLE'S EXPOSITION AND CRITICISM.

The purposes of Zeno's arguments, as set forth by Cousin, Grote, and P. Tannery differed from the purpose as it was understood by Plato, Aristotle, and the later Greek writers. The three modern interpreters gave to the arguments settings in the history of Greek thought which exhibit them as flawless in logical rigor. On the other hand, Aristotle and subsequent Greek writers interpreted the arguments as fallacies and expended their mental acumen in attempts to point out the real nature of the fallacies. Except for the recent studies of Cousin, Grote, and Tannery, the history of Zeno's arguments against motion has been for over 2,000 years the history of attempts to explain Zeno's "fallacies." Aristotle acknowledged the great difficulty in exposing their hidden source of logical error. The sixth book of his *Physics* is devoted to the exposition of the subtle notions of continuity and infinity. In fact, the entire *Physics*, of the size of about 225 ordinary modern pages of print, gives all of its eight books to discussions of the notions of motion, divisibility, continuity, infinity, and the vacuum. As one reads the *Physics* one is impressed by the fact that the great Stagirite did not hesitate to restate his arguments repeatedly. For twenty centuries Aristotle's criticisms of Zeno have been the starting points of philosophic discussion; for twenty centuries Zeno's fallacies have puzzled many of the best minds. Like the problems of the trisection of an angle, of the squaring of the circle, and of the duplication of the cube, Zeno's fallacies have challenged some of the best brains; but, like the former problems, have finally been forced to surrender their secrets and to enter the group of "problems of the past."

Before entering upon the study of Aristotle we must refer to the atomistic

theory advanced by Leucippus (about 500 B. C.) and developed further by Democritus (about 460 to about 357 B. C.) and others. They conceived magnitudes as composed of indivisible elements in finite number. This view must have gained some ascendancy among a few early mathematicians. Democritus himself was not without mathematical ability. From a notice of Plutarch, Democritus raised the following question:[1]

"If a cone were cut by a plane parallel to its base,[2] what must we think of the surfaces of the sections, that they are equal or unequal? For if they are unequal, they will show the cone to be irregular, as having many indentations like steps, and unevennesses; and if they are equal the sections will be equal, and the cone will appear to have the property of a cylinder, viz., to be composed of equal, and not unequal, circles, which is very absurd."

This paradox of Democritus places difficulties in the way of accepting the notion of an infinitesimal and thereby indirectly favors the idea of divisibility in only a finite number of parts.

In Antiphon's attempt to square the circle it is assumed that straight and curved lines are ultimately reducible to the same indivisible elements. Antiphon, according to the testimony of Simplicius and Philoponus, inscribed in a circle a square and, by arc-bisection, obtained regular polygons of 8, 16, 32 sides, and so on. He assumed that a polygon could be reached that coincides with the circle. Simplicius observes:[3]

"The conclusion here is manifestly contrary to geometrical principles, not, as Alexander maintains, because the geometer supposes as a principle that a circle can touch a straight line in one point only, and Antiphon sets this aside; for the geometer does not suppose this, but proves it. It would be better to say that it is a principle that a straight line cannot coincide with a circumference, for one without meets the circle in one point only, one within in two points, and not more, and the meeting takes place in single points. Yet, by continually bisecting the space between the chord and the arc, it will never be exhausted, nor shall we ever reach the circumference of the circle, even though the cutting should be continued *ad infinitum:* if we did, a geometrical principle would be set aside, which lays down that magnitudes are divisible *ad infinitum*."[4]

Aristotle's idea of continuity differs from the idea of continuity as developed by Georg Cantor and his followers. Aristotle's is a sensuous, physical continuum, in which there is an intimate bond between its elements; Cantor's is a collection of elements, arranged in order, infinite in number, but externally to each other; it is purely abstract and transcends the power of the imagination to grasp it. As Aristotle's *Physics* is not available in English translation, we shall translate from it rather freely. Says Aristotle:[5]

"If things continuous, things touching one another, and things successive are as described above, namely that continuous things are those whose extreme limits are one [united], things

[1] Plutarch, *de Comm. Not.*, Vol. IV, p. 1321, ed. Didot. Quoted from Allman, *Greek Geometry*, 1889, p. 81.

[2] This passage obviously means, that the cutting plane is infinitely near to the base of the cone.

[3] *Simplicii comment. in octo Aristotelis physicae auscultationis libros.* Venetiis, 1526, § 80. We quote the translation given in G. J. Allman's *Greek Geometry*, 1889, p. 66.

[4] According to another commentator, Themistius (317–387 A. D.), Antiphon began his process of squaring the circle, not by inscribing a square, but by inscribing an equilateral triangle.

[5] Aristotle's *Acht Bücher der Physik*. Griechisch und Deutsch von Carl Prantl, Leipzig, 1854, p. 279, Buch VI, § 1.

touching are those whose extreme limits have the same position, things successive are those between which there lies nothing that is like them, then it is impossible that a thing continuous consist of indivisible things, as for instance, that a line be made up of points, the line being taken as continuous, and a point as indivisible."

This passage is made clearer by a quotation or two from the *Physics*, Bk. V, § 3:

"Continuous are those things which in their union become one in nature; and just as that which holds them in continuity together becomes one, so the whole will be one, as for instance by means of a nail, or glue, or an adherence or an accretion."

"Things continuous must necessarily touch; on the other hand, things touching are not necessarily continuous, for the extreme ends, even though locally together, are not necessarily one and the same."

Returning to the *Physics*, Bk. VI, § 1, we reproduce the chief part of Aristotle's argument which leads him to the conclusion that a line is not made up of points:

"For neither are the extreme ends of points one [united], for the reason that, of an indivisible, the one cannot be an extreme and the other another part, nor are the extreme ends locally together, since the indivisible can have no extreme."

According to Aristotle, things continuous are always divisible into parts that are continuous. In the same way, time is not made up of parts considered as indivisible *Nows*.[1] We quote from *Physics*, Bk. VI, § 2:

"As each magnitude is divisible into magnitudes (for it has just been proved that the continuous cannot consist of the indivisible, and every magnitude is continuous), the faster of two must necessarily in the same time be moved through a larger distance and in a less time through an equal distance, and also in a less time through a larger distance, as indeed some define the faster. . . ."

"Since all motion takes place in time, and in each time it is possible for something to be moved and everything movable can be moved faster and also slower, then there can occur in each time the faster and the slower motion. If this is so, then time must be continuous; by the continuous I mean that which is always divisible into divisible parts. . . ."

"If time is continuous, so is distance, for in half the time a thing passes over half the distance, and, in general, in the smaller time the smaller distance, for time and distance have the same divisions; and if one of the two is unlimited, so is the other. . . . For that reason the argument of Zeno assumes an untruth, that one unlimited cannot travel over another unlimited along its own parts, or touch such an unlimited, in a finite time; for length as well as time and, in general, everything continuous, may be considered unlimited in a double sense, namely, according to the [number of] divisions or according to the [distances between the] outermost ends."

This profound criticism directed against Zeno refers evidently to the "Achilles" where we are in danger of modifying, in our minds, the conditions actually existing; we are in danger of thinking of the distance between Achilles and the tortoise as decreasing through an unlimited number of subdivisions, while the time for traversing these successive subdivisions is thought of as about the same for each. There results from this distortion of events an unlimited subdivision of a finite distance, and an unlimited accumulation of finite time-intervals; the former yields what from a modern view-point constitutes a convergent geometric series of space-intervals; the latter a divergent series of time-intervals. A finite distance is made to be traversed in an infinite time.

Aristotle's arguments against the "Achilles" and the "Dichotomy" are the

[1] See *Physics*, Bk. VI, § 9, quoted earlier in this article.

same. He touches upon them in other parts of his *Physics*. Since a line cannot be built up from points, a line cannot actually be subdivided into points.[1]

"The continued bisection of a quantity is unlimited, so that the unlimited exists potentially, but is actually never reached."[2]

"Previously we refuted this [the "Dichotomy"] by the fact that time has unlimitedly many parts, in consequence of which there is no absurdity in the consideration that in unlimited time-intervals one passes over unlimitedly many spaces . . . [but as Aristotle does not consider this a full explanation, he continues:] If one divides a continuous line into halves, he uses one point at two, for he makes it the beginning [of one part] and the end [of the other]; in this manner proceeds he who . . . bisects, but in this division neither the line nor the motion are continuous; for . . . in the continuous there are, to be sure, unlimitedly many halves, but not actually unlimited, only potentially so."[3]

Of the numerous passages in Aristotle's *Physics* which might be quoted as bearing on the "Arrow" we choose the following (Bk. VI, § 8):

". . . A thing is at rest, when it is unchanged in one *Now* and still in another *Now*, it itself as well as its parts remaining in the same status. . . . There is no motion nor rest in the *Now* . . . In a time-interval, on the contrary, it [a variable] cannot exist in the same state of rest, for otherwise it would follow that the thing in motion is at rest."

Aristotle's disproof of the "Stade" is given at the end of our first extract from the *Physics* (Bk. VI, § 9), where it is pointed out that Zeno falsely assumes a body to move with the same velocity relative to a body that is at rest, as it would with respect to one in motion.

Further Comments on Aristotle. The main criticism to be made on the profound arguments of Aristotle is that they do not go far enough. The "Dichotomy" and "Achilles" involve the theory of limits, a theory which in very recent time has been found to be in need of reconstruction and which was imperfectly developed in Greek antiquity. Particularly insistent in our mind is the query raised by Zeno's arguments, *how is it possible for a variable to reach its limit?* This query finds no reply in Aristotle. To be noted is also the fact that Aristotle denied the existence of actual infinity, as distinguished from potential infinity. The "Arrow" called for a sharp definition of the *Now* (the instant). Has the instant no duration? Is it, so to speak, a point of time? Or is the instant an infinitesimal—some constant different from zero, yet smaller than any finite quantity? Here Aristotle drew a sharp line; the *Now* was a point of time; it had no duration; in the *Now* there could be neither motion nor rest.

It will be seen that the mathematical concepts herein involved are most fundamental. The concept of a limit involves notions of infinity and is historically connected with the concept of the infinitesimal. All these concepts are basic. It would be difficult to select three other notions more far-reaching in mathematical science than *infinitesimal, infinity, limit.*

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[1] Aristotle, *Physik*, VI, 1, Prantl's ed., p. 281.

[2] Aristotle, *Physik*, III, 7, Prantl's ed., p. 141.

[3] Aristotle, *Physik*, VIII, 8, Prantl's ed., pp. 445, 447. See also Aristotle's *Lib de lineis insecab.*, p. 968.

III.

C. A Two-Thousand Year Struggle for Light.

1. The Greeks after Aristotle.

Whether Plato and Aristotle correctly explained the nature of Zeno's arguments and the purpose which Zeno himself had in mind in presenting them, is of no concern in tracing the history of thought on this subject after the time of Aristotle. Apparently the writings of Plato and Aristotle constituted the sources of information for later writers. Zeno's arguments were given by Aristotle in the form of "fallacies," and it is the influence of these "fallacies" that remains to be traced. There is little doubt that this influence was great upon the development of Greek geometry. Since Aristotle, with all his dialectical skill, was not able to satisfactorily explain all the paradoxes which had arisen in the study of the infinite and of motion, the conclusion of H. Hankel[1] and other recent historians of mathematics is probably correct, that the infinite and infinitesimal were banished from the classic Greek geometry for the sake of greater rigor. We shall see that recently discovered manuscripts of Archimedes confirm this view. Mathematicians assumed that every magnitude is divisible at pleasure. The doctrine of incommensurable lines rests upon the possibility of unlimited divisibility. The denial of the existence of the infinitesimal goes back to Zeno who is reported by Simplicius[2] to have stated: "That which, being added to another, does not make it greater, and being taken away from another does not make it less, is nothing." This momentous question presented itself twenty-two centuries later to Leibniz who gave different answers. In one exposition Leibniz extended the definition of equality so as to declare magnitudes as equal when they differ from one another by an incomparably small quantity. The later Greek mathematicians followed a radical policy toward the infinitesimal; they formally excluded it from demonstrative geometry by a postulate. This was done by Eudoxus (408–355 B. C.), by Euclid (about 300 B. C.) and by Archimedes (287–212 B. C.). Archimedes gives the postulate, which he attributes to Eudoxus, as follows:[3]

"When two spaces are unequal, it is possible to add to itself the difference by which the lesser is surpassed by the greater, so often that every finite space will be exceeded."

Euclid in his *Elements* (Bk. V, Def. 4) gives the postulate in the form of a definition:

"Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other."

The Method of Archimedes, a book formerly thought to be irretrievably lost, but fortunately discovered by Heiberg in 1906 in Constantinople, gives inter-

[1] H. Hankel, *Geschichte der Mathematik im Alterthum und Mittelalter*, Leipzig, 1874, p. 120.

[2] Simplicius, *Phys.* 30*a*. Quoted by E. Zeller, *History of Greek Philosophy*, Vol. 1, London, 1881, p. 615.

[3] Archimedes, *De quadr. parabol. Praef.*

esting evidence that the notion of infinitesimals, though not used by Archimedes in formal demonstrations, was employed by him in tentative research as a method of discovery. He considered infinitesimals sufficiently scientific to suggest the truths of theorems, but not to furnish rigorous proofs. The process is mechanical, consisting of the weighing of infinitesimal elements, which he calls straight lines or plane areas, but which are really infinitely narrow strips or infinitely thin plane laminæ.[1] The breadth or thickness is regarded as being the same in the elements weighed at any one time.

Further evidence that infinitesimals lingered in the minds of Greek thinkers through the centuries is furnished by the skeptic, Sextus Empiricus (200 A. D.), who advances the paradox that, when a line rotating in a plane about one of its ends describes a circle with each of its points, these concentric circles are of unequal area, yet each circle must be equal to the neighboring circle which it touches.[2] The difficulty encountered here is similar to that raised by Democritus over 500 years earlier.

An interesting remark about Zeno is made by Plutarch (about 90 A. D.) in his life of Pericles. He says that Pericles was also a hearer of Zeno, the Eleatic, who "also perfected himself in an art of his own for refuting and silencing opponents in argument; as Timon of Phlius describes it—

> Also the two-edged tongue of mighty Zeno, who,
> Say what one would, could argue it untrue."

Thus we see that the personality of Zeno and some of the ideas involved in his "paradoxes" appear here and there on the surface of Greek thought, thereby indicating an underground flow of those ideas down the centuries of Greek history.

Sextus Empiricus also gives a version of the "Arrow" much like what we quoted from Diogenes Laertius. He attributes the paradox not to Zeno but to Diodorus Cronus. The existence of motion is disproved thus: If matter moves, it is either in the place in which it is, or the place in which it is not; but it cannot move in the place in which it is, and certainly not in the place in which it is not: hence it cannot move at all. To this Sextus Empiricus replies by stating another argument equally paradoxical and therefore far from illuminating: By the same rule men never die, for if a man die, it must either be at a time when he is alive, or at a time when he is not alive; hence he never dies.

2. The Romans.

The subtleties of Zeno's arguments on motion attracted little attention among the Romans. Lucretius (96–55 B. C.) used the notion of infinity in arguments on atomic theory. He reasoned that one must assume the existence of atoms (indivisible, but not mathematical points), otherwise each body,

[1] T. L. Heath, *Method of Archimedes*, Cambridge, 1912, p. 8.

[2] Sextus Empiricus, *Adv. math.*, I, III, § 66 ff., ed. Fabricius. p. 322; referred to in K. Lasswitz, *Geschichte der Atomistik* I, Hamburg u. Leipzig, 1890, p. 148.

whether large or small, would consist of an infinite number of parts, and there would be no difference between the largest and the smallest, both being infinity.[1]

Cicero and Seneca mention Zeno in passing: the one to display Zeno's love for argument even though faulty, the other to display Zeno's skepticism. Cicero[2] attributes to Zeno the following pointed syllogism: "That which exercises reason is more excellent than that which does not exercise reason; there is nothing more excellent than the universe; therefore the universe exercises reason." Seneca[3] exhibits Zeno as denying not only plurality as did his master, but denying also unity and the real existence of external objects. We have no good reason for accepting either Cicero's or Seneca's view of Zeno's aim and purpose.

3. Medieval Times.

The earliest church father known to interest himself in arguments on motion was St. Augustine (354–430 A. D.). In a dialogue on the question, whether or not the mind of man moves when the body moves, and travels with the body, he is led to a definition of motion, in which he displays some levity. It has been said of scholasticism that it had no sense of humor. Hardly does this apply to St. Augustine in his discussion of the impossibility of motion. He says:

> "When this discourse was concluded, a boy came running from the house to call us to dinner. I then remarked that this boy compels us not only to define motion, but to see it before our very eyes. So let us go, and pass from this place to another; for that is, if I am not mistaken, nothing else than motion."[4]

St. Augustine deserves also the credit of having accepted the existence of the *actually infinite* and to have recognized it as being, not a variable, but a constant. He recognized *all* finite positive integers as an infinity of that type.[5] On this point St. Augustine occupied a radically different and more advanced position than his forerunner, Origen of Alexandria, who took a decided stand against the actually infinite and supported his position by arguments which G. Cantor admits to be the profoundest ever advanced against the actually infinite.[6] In their completest form, these arguments were given many centuries later by the great Italian philosopher of the Middle Ages, Thomas Aquinas (1225(?)–1274).[7] Of importance to us is the nature of the continuum, particularly the linear continuum, as described by Aquinas. It was conceived as *potentially* divisible to infinity, since practically the divisions could not be carried out to infinity. There was, therefore, no minimum line. On the other hand, the point is not a constituent part of a line, since it does not possess the property of infinite divisi-

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[1] Lucretius, *De rerum natura*, ed. I. Bernays, Leipzig, 1886, I, 615 ff.

[2] *De natura deorum*, Book III, IX.

[3] Epistola 88.

[4] St. Augustine, *De ordine*, II, VI, 18.

[5] S. Augustin, *De civitate Dei*, lib. XII, cap. 19. The chapter is quoted by G. Cantor in "Mitteilungen zur Lehre vom Transfiniten," *Zeitschr. f. Philosophie u. philosoph. Kritik*, Bd. 91, p. 81. Separatabdruck, p. 32.

[6] G. Cantor, *op. cit.*, Separatabdruck, p. 35.

[7] Thomas Aquinas, *Summa theol.*, I, q. 7 a. 4. Quoted by G. Cantor, *op. cit.*, Separatabdruck, p. 36.

bility that parts of a line possess, nor can the continuum be constructed out of points. However, a point by its motion has the capacity of generating a line.[1]

This concept of the continuum, as held by Aquinas, is a fair representation of the prevalent medieval scholastic views on this topic. It held a firm ascendancy over the ancient atomistic doctrine which assumed matter to be composed of very small, indivisible particles, possessing the properties of matter itself. No continuum superior to it was created before the nineteenth century.

In his commentaries on Aristotle's *Physics*, Aquinas[2] explains at some length the arguments of Zeno against motion as they are given by Aristotle. Aquinas shows a complete mastery of the subject as expounded by the Stagirite, but hardly presents any new points of view.[3]

Early English Writers. The earliest Englishman known to have written on continuity and infinity is Roger Bacon (1214(?)–1294), the seven-hundredth anniversary of whose birth was celebrated at Oxford in 1914. Bacon argued against the composition of the continuum of *indivisible* parts (different from points), by renewing the arguments presented by the Greeks and the early Arabs. Bacon held that the hypothesis of indivisible parts of uniform size would make the diagonal of a square commensurable with a side; if the ends of an indivisible part of a circle are connected by radii with the center of the circle, then the two radii would intercept an arc on a concentric circle of smaller radius. From this it would follow that the inner circle is of the same length as the outer circle. This is impossible. Bacon argued also against infinity. If time were infinite, it would follow that the part is equal to the whole—a deduction which he considered absurd. Similar arguments lead him to conclude that the world is finite.[4]

The views of Roger Bacon became known more widely through Duns Scotus (1265–1308), the theological and philosophical opponent of Thomas Aquinas. However, Scotus and Aquinas took the same ground in teaching that in the continuum there existed actual, indivisible points. Thereby it is not admitted that the continuum is made up of, or consists wholly of, points; the indivisible

[1] C. R. Wallner, in *Bibliotheca mathematica*, 3. F., Bd. IV, 1903, pp. 29, 30, gives quotations from Thomas Aquinas, *Opuscula omnia*, 1562, o. 52, p. 369; o. 36, c. 2; o. 44, c. 1; o. 44, c. 2, p. 280.

[2] *Opera omnia*, Tom. II, Pars prima: *Sancti Thomae aqvinatis ex ordine praedicatorum quinti ecclesiae doctoris angelici praeclarissima commentaria in octo Physicorum Aristotelis libros. . . . Ad haec accessit Roberti Linconiensis in eosdem summa.* Parisiis, MDCLX, Lectio XI, pp. 233–237, 352.

[3] A summary of what is given in each of Aristotle's eight books on Physics is given by Robert Grosseteste (1175(?)–1253), bishop of Lincoln, whom Roger Bacon praises as a scientist of high rank. The esteem in which his writings were held appears from the fact that his summary of Aristotle is reproduced four centuries after his death in an edition of Aquinas. Of Zeno's arguments Grosseteste says (page 352): *Ad primum dicitur (sicut prius dictum est id dubitatione præcedenti) scilicet quòd continuum est infinitum secundum potentiam & tale potest transiri. Et sic patet ad secundam rationem. Ad tertium dicitur quòd Zeno dixit ipsum tempus componi ex instantibus, quod non est verum, ideò nec motus nec quies est in instanti, sed in tempore [sicut dixit Philosophus] ideò mobile non est spacio sibi aequali nisi tantum in instanti. Et cum dicitur, aut movetur, aut quiescit, negatur propositio, quia habet veritatem de es quod est in aliquo in quo aptum natum moveri aut quiescere pro tali mensura temporis.*

[4] See *Opera hactenus inedita Baconi*, Fasc. 1, Metaphysica. Edidit Robert Steele. London, p. 11; Jonas Cohn, *Geschichte des Unendlichkeitsproblems*, Leipzig, 1896, pp. 76, 77; K. Lasswitz, *op. cit.*, Vol. I, pp. 193, 195.

points might, for instance, be simply end points. These contentions are directed against the atomists. The arguments are wanting in explicitness and precision. What we said of Aquinas's commentary on Zeno applies also to Duns Scotus.[1] He gives detailed elaborations of Aristotle without offering new explanations of Zeno's puzzles. In place of Achilles and the tortoise he introduces the more familiar travelers, the horse and the ant. His commentaries are annotated by the Franciscan theologian Franciscus de Pitigianis of Arezzo in Italy, who wrote the latter part of the sixteenth century. This annotator expresses himself in favor of the admission of the actual infinity to explain the "Dichotomy" and the "Achilles," but fails to adequately elaborate the subject. Scholastic ideas on infinity and the continuum find expression in the writings of Bradwardine, the English *doctor profundus*. He says that five explanations have been given of the nature of the continuum.[2]

GROUPS OF SUBTRACTION AND DIVISION WITH RESPECT TO A MODULUS.

By G. A. MILLER, University of Illinois.

Certain kinds of groups of subtraction and division were explained by the present writer in two articles entitled: "Groups of the fundamental operations of arithmetic" and "Groups of subtraction and division." These articles were published respectively in the *Annals of Mathematics*, volume 6 (1905), page 41; and in the *Quarterly Journal of Mathematics*, volume 37 (1906), page 80. The present article is devoted to more elementary considerations, and has for its main object to exhibit interesting elementary relations between certain groups of subtraction and division, and the corresponding groups of addition and multiplication.

It is well known, and also evident, that the first $m - 1$ natural numbers together with zero constitute the cyclic group of order m with respect to addition when the sums are replaced by their least positive residues, or by zero, modulo m. That is, if in the series of numbers

$$0,\ 1,\ 2,\ \cdots,\ m-1$$

each number is replaced by itself increased by α, mod m, where $0 \leqq \alpha \leqq m - 1$, there results a certain substitution on these m numbers, and the totality of the distinct substitutions which can be constructed in this manner constitutes the cyclic group of order m. The order of the substitution corresponding to α is

[1] Duns Scoti, *Opera Omnia*, T. II: Joannis Duns Scoti Doctoris Subtilis, ordinis minorum, in VIII libros Physicorum Aristotelis Quæstiones, cum annotationibus R. P. F. Francisci Pitigiani arretini, etc. Lvgdvni, MDCXXXIX, Quaestio X, pp. 390–393.

[2] See Maximilian Curtze on the "Tractatus de continuo Bradwardini" in *Zeitschrift f. math. u. Phys.*, XIII Jahrg., Suppl., 1868, Leipzig, p. 88.

evidently the quotient obtained by dividing m by the highest common factor of α and m. In particular, when $\alpha = 0$ this substitution reduces to the identity.

If each of the numbers of the given series is replaced by itself decreased by α, mod m, there results again a substitution, and the totality of the distinct substitutions which can be obtained in this manner constitutes again the cyclic group of order m. In fact, the substitution which corresponds to α when α is subtracted from each of the given numbers is the inverse of the substitution which corresponds to α when it is added to each of these numbers, since the successive performance of these two operations leaves each of the given numbers unchanged. As an automorphism of any abelian group can be established by letting each operator of this group correspond to its inverse, it results that a simple isomorphism between the given group of addition and the given group of subtraction can be established in such a way that the operations which result from the same number in the processes of addition and subtraction correspond in this simple isomorphism.

The simple isomorphism which has been considered exhibits very clearly that the operations of subtracting successively each of the numbers of the given series from every number of this series have the same relative properties as those of adding these same numbers to every number of this series. The fact that each operator corresponds to its inverse in the given simple isomorphism may be regarded as an extension of the concept that addition and subtraction are inverse operations. In fact, not only are these operators inverses but the two groups which they constitute under the given conditions are such that every operator of the one corresponds to the inverse of the other.

When the $\phi(m)$ natural numbers which do not exceed m and are prime to m are combined by multiplication, mod m, they form one of the most important classes of abelian groups. For any particular value of m, the corresponding group of order $\varphi(m)$ can be obtained by replacing these $\phi(m)$ numbers by the set obtained by multiplying each of them by k, provided k is prime to m and mod m is replaced by mod km. In fact, it has been observed that a necessary and sufficient condition that a series of distinct natural numbers constitutes a group as regards multiplication, mod m, is that each of the numbers of the series has the same highest common factor with m and that the quotient obtained by dividing m by this highest common factor is prime to this factor.[1] The special but fundamental case when this highest common factor is unity is the one which is generally treated in the text-books.

In this special case it is very easy to see that the same group of order $\varphi(m)$ results if the operation of multiplication is replaced by that of division, when the quotient α/β, mod m, is defined as usual, as an integer γ such that $\beta\gamma \equiv \alpha$, mod m, α and β being natural numbers. The substitution on the given $\varphi(m)$ numbers which results if all of these numbers are multiplied by any one of them, for instance α, is clearly the inverse of the one which results when all of these numbers

[1] *Annals of Mathematics*, series 2, vol. 6 (1905), p. 44. It may be observed that the statement of this theorem in the *American Journal of Mathematics*, vol. 27 (1905), p. 315, is inaccurate.

are divided by α. Hence the two substitutions which result from operating with the same number correspond in one of the possible simple isomorphisms between the given groups of multiplication and division, mod m.

In the more general case noted above, when a series of distinct natural numbers constitute a group as regards multiplication, we cannot always pass directly to a group of division without further restrictions. In fact, this more general set of numbers does not include unity with respect to the modulus, while the division of a number by itself gives unity for a quotient. It is, however, easy to see that whenever a set of numbers constitutes a group as regards multiplication, mod n, these numbers must also constitute a group as regards division, mod n, provided the quotient is restricted to the given set of numbers mod n.

As the converse of this proposition is evidently also true it results that *a necessary and sufficient condition that a set of distinct natural numbers constitutes a group with respect to division, mod m, when all of these numbers are divided successively by each one of them and the quotients are restricted to numbers of the set, is that m has the same highest common factor with each of these numbers and that the quotient obtained by dividing m by this factor is prime to this factor.* For instance, the set of numbers; 2, 4, 8, 10, 14, 16 constitutes the cyclic group of order 6 with respect to each of the operations of multiplication and division mod 18, provided that in the latter case the quotients are restricted to this set of numbers. In general, the groups of multiplication and division which are obtained in this manner are simply isomorphic, and the substitutions which correspond to the same number in these two groups are the inverses of each other.

CALIFORNIA TEACHERS OF MATHEMATICS.

The following extracts from the report of the last annual meeting of the Mathematics Section of the California High School Teachers' Association by the chairman, Professor Henry W. Stager, of Fresno Junior College, are significant in many ways, and will be of interest to readers of the MONTHLY:

At the annual meeting in July, 1914, the Mathematics Section of the California High School Teachers' Association adopted an official reading course for the present school year. The purpose of this course is to arouse a greater interest in the subject on the part of teachers of mathematics rather than to increase their knowledge of mere mechanical methods of presentation. Every teacher is urged to undertake the careful reading of one or more of these books this year. The course is divided into three sections, graded according to difficulty. Each teacher can find some book suited to his individual needs. The University of California will grant credit for the study of certain of the books as part of the work in university extension. Detailed information may be obtained by addressing the University Extension Division, University of California, and referring to the course herewith.

Teachers are urged to ask their trustees to place the entire list in the school

library at the earliest possible opportunity. The list is also suitable for public libraries,—in fact it forms a most excellent list for the general reader who is interested in mathematics and desires an introduction to the modern view-point. Teachers are asked to suggest the purchase of these books by the local public libraries. The list which follows has been recommended only after the most careful consideration by the committee in charge.

SECTION I. Books of easy grade.

1. Ball: *A Primer of the History of Mathematics.*
2. Cajori: *A History of Mathematics.*
3. Smith-Karpinski: *The Hindu-Arabic Numerals.*
4. Whitehead: *An Introduction to Mathematics.*

SECTION II. Books of medium grade.

5. Ball: *Mathematical Recreations and Essays.*
6. Beman and Smith: *Klein's Famous Problems in Elementary Geometry.*
7. J. W. Young: *Fundamental Concepts of Algebra and Geometry.*
8. Fine: *The Number-System of Algebra.*

SECTION III. Books of more advanced grade.

9. J. W. A. Young, Editor: *Monographs on Modern Mathematics.*

In connection with the above, the committee suggest the reading of a good mathematical journal and recommend for this purpose, THE AMERICAN MATHEMATICAL MONTHLY.

In his report, the chairman stated that he had sent out two letters to each of the 400 teachers of mathematics in the state, calling their attention to the study plan adopted by the Section in 1913. The work of the Section had also been presented at the four sectional meetings of the California Teachers' Association. As far as could be judged from correspondence and interviews with teachers, there seemed to be a new and increased interest in the better teaching of mathematics throughout the state. A message from Professor H. E. Slaught, managing editor of THE AMERICAN MATHEMATICAL MONTHLY, was read:

> "Press on in the way you have started. Show the country what can be done by a body of teachers who have ambition to move *onward and upward*, who believe that the best enrichment of the secondary field depends upon the strongest and highest development of the teachers themselves, not simply through activity on the *dead level of daily routine*, but also, and most emphatically, through personal power gained in the higher ranges of thought and study. We are just beginning to realize the importance of this matter in this country, though other countries, and especially Germany, have given us object lessons in plenty. But this is part of the general uplift in our whole educational system and it will spread rapidly when it is once understood. In fact, this is the only explanation of the wonderful growth of the summer school idea in the universities. Thousands of teachers are resolving to take part in the uplift by *uplifting themselves through higher study and training*. To be in the lead in this great movement is no small honor. Hence, God speed the California teachers in this "forward look."

The chairman advocated a continuous policy in the work of the Section, especially for the coming year. To this end the previous committee on ways and means was reappointed, consisting of Professor D. N. Lehmer, chairman, Miss Thirmuthis Brookman, of San Mateo, and Miss Sadie L. Gilmore, of Colusa, together with the chairman of the Section.

Two of the papers read concerned the administration of the intermediate schools, sometimes called junior high schools, consisting of the seventh, eighth, and ninth grades.

Miss Thirmuthis Brookman, for some years head of the mathematics department of the Berkeley high schools, insisted that all mathematics must be correlated with the life of the students. She illustrated her principles by applications of the arithmetic of investment and expenditure, civic arithmetic, and the like, as worked out in Berkeley for the first two years of the intermediate school. The last year was devoted to algebra, which was made extremely practical, by the alternation of the work on the abstract principles of algebra with chapters dealing with the applications of proportion, such as belted pulleys in sewing machines, gears in mesh, levers, etc. The natural focus of the ninth-grade work was the mastery of the quadratic formula.

Mr. Will C. Wood, California commissioner of secondary schools, showed that mathematics should have a place in this curriculum, from both a *practical* and a *disciplinary* standpoint. Mathematics should be required throughout the three years, leading up to a mastery of arithmetic and the beginning of algebra in the eighth year. The ninth year should complete the essentials of elementary algebra and include some of the fundamental ideas of geometry, such as parallel lines, similarity, and simple geometric constructions, all woven together into a unified whole. The final topic which he considered was the preparation and certification of the teachers for this school. A special preparation will be necessary for this work, involving more time and study than that required for elementary school teachers, but not so highly specialized as the teacher of the secondary school. New methods of teaching should be effected in which the *individual* is the aim of the teacher's thought rather than *uniformity*.

Professor D. N. Lehmer, of the University of California, spoke with great effect on "How shall the isolated teacher of mathematics keep up his interest." He showed how mathematics, more than most other branches, could be a constant source of inspiration under any conditions, because neither laboratories nor immense libraries are essential in the search after her truths. He advised that teachers set apart a definite amount of time each week, even if it be only a few hours, and devote it to quiet and earnest consideration of some mathematical subject. After suggesting some topics for such study, he added that reviews of previous work are often exceedingly profitable. This address was full of inspiration and encouragement for every teacher.

Professor H. W. Marsh, of Pratt Institute, Brooklyn, gave a brief resumé of the educational ideals of the past and then showed how the modern ideal of educating each child so as to render him of greatest service to society is a transition in *name only* from the cultural ideal so long predominant. The main effect of the cultural plan has been to drive a very large percentage of our children from school by the end of the fifth grade. Many attempts have been made to introduce problems of real life into mathematics, but these problems have been largely of an artificial type. The solution of this question can only be brought about by pre-

senting the fundamental propositions and principles of mathematics in such constructive and developmental form as to *make each student feel the joy of individual discovery and creation.* The address contained some specific suggestions as to the subject matter and method of presentation of mathematics to meet these new ideals.

Professor Stager emphasized the need of greater coöperation between the university and the secondary school. This can be shown by a sympathetic interest on the part of the former in the varied, and often difficult, problems of the latter. High standards should be set by the university and should be insisted upon. Helpful suggestions should be offered through visitation of university instructors to the schools and teachers' meetings, and through carefully prepared bulletins. The university entrance requirements should receive re-adjustment: (1) by defining the two years of required mathematics as *elementary mathematics,* rather than as *algebra and plane geometry,*[1] which would permit of greater efficiency in teaching and afford those teachers favorable to the idea of so-called "unified mathematics" an opportunity to work out their ideas; (2) by allowing the second year of algebra to be devoted, one half to the more important, but less theoretical portions, of the present requirement, and the other half to the elementary fundamental principles of analytical geometry, more especially of the straight line and the circle. This would prove equally advantageous to both the university and the secondary school.

The two sessions were each attended by about seventy-five teachers and the papers were followed by spirited discussions.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY, University of Minnesota.

Elementary Mathematical Analysis. A text-book for first year college students. By CHARLES S. SLICHTER. McGraw-Hill Book Company, New York, 1914. xiv + 490 pages.

Note.—The editors asked two men representing quite different phases of mathematical interests to review this book, and they take pleasure in presenting to the readers of the MONTHLY these two reviews.

I.

For years it has been assumed that mathematics should be administered in definite doses. After a pupil has emerged from the grammar school he is to be treated to one year of elementary algebra, one of plane geometry, then a half year each, of solid geometry, plane trigonometry, intermediate algebra, and advanced algebra. The tendency has been to prescribe the table of contents of each of these subjects. A student who can pass examinations in them all is ready to commence the study of analytic geometry, for which considerable familiarity

[1] This provision was made by the University of Chicago two years ago. EDITOR.

with graphical methods can be presupposed from the algebra. But in many cases the preparation in algebra has been found unsatisfactory, so that now attempts are being made to accept alternate subjects for college entrance and to provide for instruction in advanced algebra in college. In the book under review a much greater step is taken; only elementary algebra and plane geometry are presupposed; the volume attempts to provide instruction in trigonometry, intermediate and advanced algebra, and in plane analytic geometry. Use is made of solid geometry but once (p. 416).

To make such a programme economical, the school algebra and geometry should be given during the last two years of high-school rather than during the first two years as at present, otherwise the student would commence his college mathematics in a very rusty condition. Partial provision is made for this state of affairs, however, by including a sixteen page review of elementary algebra in the volume.

The table of contents gives only the names of the chapters, some of which contain several pages of material not connected with the title. This lack is partly compensated for by a full index. Several hundred exercises for the student are included, the answers to which are not furnished.

The first and second chapters are largely descriptive; they deal with the ideas of scale, functionality, and of graphical representation, with particular attention to power functions. The presentation is enlivened by a large number of illustrations of practical applications. The next chapter is a successful combination of the elements of trigonometry and analytic geometry of the circle. The trigonometric functions are at once defined for the general angle, thus making unnecessary much of the explanatory matter found in the ordinary texts. Indeed, this scheme could have been applied to a number of particular cases, where the author prefers to employ the ordinary right-angle triangle. It is to be regretted that the same generality was not always preserved in the later chapter on the addition formulas.

It seems odd to find in the chapter entitled "Ellipse and Hyperbola" thegraphs of $y = \tan x$, $y = \cot x$, $y = \sec x$, and proofs of the theorems: limit $x/\sin x = 1$, limit $x/\tan x = 1$. The chapter on simultaneous equations gives the factor theorem and a number of graphical illustrations, but nothing more from the theory of equations. Synthetic division and Horner's method are no where mentioned.

Next follows a chapter on permutations and combinations, the main use of which is to prove the binomial theorem; the next chapter, on progressions, is provided with a number of graphical illustrations. The substance of these three chapters is usually included in elementary algebra.

The longest chapter (68 pages) and perhaps the best chapter is entitled "The logarithmic and trigonometric functions." It begins with a historical outline and accompanies it with graphical illustrations, making the development natural and interesting. After the elementary properties have been established, they are applied to explain the double scale, the slide rule, uses of logarithmic paper and the interpretation of data gathered from a number of engineering problems.

Chapter IX includes the machinery of trigonometry, the solution of triangles, the graph of a power of sin x, etc.

Another commendable chapter is the tenth, on waves; it discusses the meaning of the terms used and shows how to apply the analysis in a large variety of ways. The chapter on complex numbers seems unnecessarily long (40 pages). It seems to the reviewer that all the essential facts could be stated in much less space.

The chapter on loci is concerned almost entirely with problems. They include several on the lemniscate, the cycloidal curves, and various others. The chapter on the conic sections contains the traditional treatment of many of the problems of analytic geometry, including tangents and normals, the general equation of a conic, and confocal conics. The criterion given for determining the positive and negative sides of a line is not applicable to lines which pass through the origin.

No mention is made of determinants. The idea of a derivative is introduced to obtain the slope of a general parabola, and of the exponential curve. The first case is easily justified, but the second is questionable, at first sight.

Whether this programme is a wise one can be determined only from experience. A priori praise or blame has but little meaning; at any rate it is an earnest effort to remove some of the difficulties of our present schedules.

VIRGIL SNYDER.

CORNELL UNIVERSITY.

II.

The first and most permanent impression which Slichter's Elementary Analysis makes upon one is of originality, of extraordinary and all-pervasive originality. The work is so different from any I have taught or any I know that I should be unwilling to give any estimate of its usability as a class text. Better than a review would be a series of statements by a number of persons who had used the book. And in this connection I might suggest that the MONTHLY could do no greater service to collegiate teaching and teachers of mathematics than by replacing or supplementing its "Book Reviews" by conducting an experience meeting on texts which have had a fairly wide adoption.

When we review an ordinary book we have only to compare it with others of its like and to make some rather inaccurate estimate of how it would impress a class. When we have to deal with an iconoclastic work we must go back to first principles, and these principles have not yet been definitely established, except in so far as the historical development of mathematical teaching to the present time may be considered to have established them, and this test is always unfair to the original work.

We shall therefore ask first the question: What should be taught in the freshman year? The canonical answer is, algebra, trigonometry, and analytic geometry, in various proportions. Now analytic geometry, except for that part which introduces the student to coördinates and elementary curve tracing and which may better be designated as graphical algebra, is a purely mathematical

subject which is distinctly difficult for a student to assimilate in his most immature collegiate year. The real problem of analytic geometry is to obtain geometrical results by algebraic means, and experience seems to show that this is too much for the freshman; moreover, it is beyond the needs of any but the student of pure mathematics or of really advanced applications. There has sprung up a strong tendency toward the elimination of true analytics from freshman work, and we hope the tendency will become a fixed habit. Slichter pays scant attention to analytics.

The process of attrition has also been applied to trigonometry. There are many old texts built upon the practice of nearly, if not quite, a whole year's work devoted to the subject. Undoubtedly plenty of material of importance exists from which a long course in trigonometry can be constructed, but not without calling upon many items which are unnecessary for the ordinary student. We are therefore getting down to twenty or thirty lessons on trigonometry, and Slichter has no more than this (at least of the canonical sort of trigonometry).

When our students come to college they have been over a considerable amount of algebra, probably as much as, or more than, most of them will need; but they frequently do not know how to use rapidly and accurately even the smaller amount which they must meet almost daily. A review of secondary school algebra would be a considerable portion of the first few weeks work in the freshman year if we put our students' efficiency before our academic pride. Slichter closes his book with such a review and with the statement that the review should precede Chapter I. (Why not print it there?) He does not include simultaneous quadratic equations in this review; they are treated appropriately but very inadequately, especially as regards exercises for the student to do, after the common second order curves have been presented. As for additional algebra, the author gives us permutations and combinations, the binomial theorem, and progressions, interesting material briefly treated.

We see therefore that as regards algebra, trigonometry, and analytic geometry we are on safe modern ground in retaining the essentials and in eliminating the more advanced parts.

The treatment of these three subjects is according to the mixed method. We do not proceed with one until it is completed but oscillate from one to another. The first two chapters are analytics (used, as always in this review, for graphical algebra or curve tracing), the third is trigonometry, the fourth returns to curve tracing and the fifth to its more algebraic side (solution of equations). Chapters VI and VII are algebraic, Chapter VIII is on logarithms and exponentials, and Chapter IX is on trigonometry again. The tenth and eleventh chapters on waves and complex numbers (with polar diagrams of periodic functions) are for the most part curve tracing once more, while the last two chapters on loci and the conic sections approach true analytics without quite reaching it.

It will be noticed that the author has really welded his material together; he does not switch violently from one subject to another just for the sake of pursuing the mixed method; there are modulations. Ordinarily it would not

be possible for the teacher to transpose the chapters and rearrange the work according to subjects—though I am by no means certain that such a transposition could not be made with Chapter IX so that it might follow Chapter III.

It is now time for a second fundamental question: Is the mixed method good? We all know the difficulty it is supposed to avoid, namely, the isolation of the student's knowledge into mutually insulated bodies. Some also feel that by not classifying a student's knowledge you give him greater power, you lead him to apply all his mathematical equipment to a new problem, instead of some particular part of his equipment such as trigonometry. The mixed method as contrasted with the compartment method of instruction is so young that we cannot yet say whether it is better or worse. There is the possibility, which at times appears to me as a strong probability, that by introducing broad connecting and conducting links between the different compartments you so reduce the potential that the efficiency is lowered. May it not be true that most problems which the majority of us solve are really problems in algebra, in trigonometry, or in analytics, and that the training in classification according to these topics is highly valuable? Is not the mixed method mixing to the student, and may not the oscillation be allied to vacillation? Only experience can answer.

As the author has reduced the ordinary subjects to such moderate compass, what has he done with the rest of his five hundred pages? He has introduced a large number of topics not usually found in a freshman text, perhaps not in any mathematical text used by mathematicians—double and triple scales, the slide-rule, plotting on logarithmic and semi-logarithmic and sinusoidal coördinate paper, simple harmonic motion, damped vibrations, logarithmic decrement, waves, shearing motion, graphs of many engineering formulas, and plenty of odds and ends. In these and in many other ways the work suggests Perry's "Calculus for Engineers."

It is true that all the things which the author takes up may logically be taken up; but we are brought back to the first question: What ought to go into the freshman course? For myself I do not believe in a mathematical or engineering museum for first year students. When a book starts in to give a series of illustrations and appends a footnote (as on page 68) stating: "The instructor is expected fully to explain the meaning of the technical terms here used," I get out of sorts with the author. The freshman is an immature person with a limited range of ideas and with a great many ideas in this limited range in need of clarification; the fewer unfamiliar terms that are introduced in his instruction the better, and the greater the chance of impressing on him the main thing in mathematical instruction, to wit, to think clearly.

Among the ideas which every freshman has, but has only vaguely, is the idea of speed. The word speed does not occur in Slichter's index, nor does velocity except for (uniform) angular velocity and velocity of a wave. The word rate is also missing. One may remark that without entering upon the notions of calculus these words cannot be defined. Precisely, and that is the fundamental reason for entering as promptly as possible upon those ideas. The author urges

that one of his chief aims is to impress the concept of functionality on the student. He does impress it so far as graphical algebra alone can impress it; but of prime importance in the idea of functionality is that of a rate of change, and into this he does not go except for the mention once or twice of slope.

Here is the really immoral element in the book: The author has cut down the algebra, the trigonometry, and the analytics to a point where the student has not acquired a real facility of operation with any of the three, and then instead of turning the time and space thus saved into a presentation of the elements of the calculus, he has frittered both away with frills which are more or less unintelligible to the student and of incomparably less value to him if understood.

There are three objects in mathematical instruction: (1) To give the student a training in formal manipulation which will suffice him for his ordinary needs—and to give this it is necessary to take him over considerably more than what he will actually use, just as to play satisfactorily a sonata one needs a technique developed beyond that of the piece; one must have reserve power. (2) To formulate for the student and to drill into him the fundamental ideas which he meets daily and with which in many cases he is already vaguely familiar—speed, rates, limits of sums—and here again he should go far enough to have reserve power, whereas most texts on calculus do not take him far enough so that he can formulate a definition for flux or induction. (3) To give him a taste of the applications of his ideas and his technique in mechanics or physics—and this could be done by the departments of mechanics and physics if they were not so terribly chary of using the mathematics that the student has (possibly the teacher has it no longer?).

Now it is axiomatic that you cannot teach all these subjects in a single year, and to freshmen at that. In particular (3) must be almost entirely omitted at first. You cannot even give a freshman both his technique and his ideas. In this dilemma the preference used to be given to qtechniue, and with a good reason; for the young will unhesitatingly accept a formal method and, being imitative rather than rational, will rapidly acquire all the necessary facility. If this preference were still to be followed, we should give the freshman a thorough course in trigonometry, including trigonometric equations and identities, which are insufficiently treated by our author, and perhaps a few applications to statics in the plane; we should follow with a small amount of curve tracing, and then finish the formal differential calculus including a few geometric and kinematic applications. The sophomore year would begin with formal integration.

The serious objection of a practical nature to this procedure is that the ideas of the calculus do not come early enough to be most useful in the work in physics. We ought therefore to turn to the other preference and give the freshman a brief course in trigonometry (which might better be given in the secondary schools), a little curve tracing, and the ideas of the differential and integral calculus, keeping the formal work so simple that it will offer no difficulty. The sophomore year would then begin with trigonometric identities and drill in differentiation and integration. The technique will not come too late for the work in physics

and engineering, because not much technique is required for the early instruction in these subjects.

We have gone into these matters in such detail because we have had to go back to fundamentals and because we believe that the author has made the worst possible use of his extra space and time. We shall now try to give some justification for our feeling that the book is intensely original. The ellipse is defined as the locus obtained by shortening the ordinates of a circle in a definite ratio, and some two hundred pages later the tangent to the ellipse is found by applying the same rule to the ordinates of the tangent to the circle. The ellipse is given in parametric form, the loci $y = \tan x$ and $y = \sec x$ are plotted, and the hyperbola is defined and plotted from its parametric form $x = a \sec \theta$, $y = b \tan \theta$. The tangent to the hyperbola is not found. The tangent to the parabola is found by expressing the fact that it cuts the curve in coincident points; that to the circle, by its perpendicularity to the radius. The normal equation of a straight line and the distance from a point to a line occur near the end of the book in the chapter on conics, presumably because by means of them the passage from the tangent to a circle to the tangent to an ellipse is made.

At the very beginning of the book the student is told what drawing materials he should have and a considerable amount of graphical computation is given. This comes in handy in treating logarithms which are introduced by arranging arithmetic and geometric progressions in parallel columns. Exponential curves are constructed and the number e is defined as that value of r for which $y = r^x$ passes through (0, 1) with the slope 1. In this connection and only here is the slope related to the limit of the slope of a secant. Later e is found by a familiar operation on the binomial expansion, an operation which is justified by the (presumably untrue) statement that in the calculus it is shown that

$$\lim_{n=\infty}\left[1+1+\frac{1-\frac{1}{n}}{2!}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!}+\cdots\right]=1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots.$$

These illustrations, taken with earlier mention of unusual things discussed by the author, may indicate the originality of the work, an originality which grows on one constantly. Indeed it would be little exaggeration to say that if there was an unusual way of doing anything, the author has chosen that way and worked it down so smoothly that few who did not know would consider it new.

For one who lays such stress on calculation and gives such elaborate tables illustrating how an expert computer would arrange the solution of a triangle, it is surprising to find the author paying no attention to the vertical alignment of his decimal points in the table. I have not seen many examples of the work of expert computers, but what few I have seen look very different in this particular. Moreover, in solving a triangle with two sides and the included angle given, why use the tangent formulas? Why not be original and drop the perpendicular from the shorter side upon the longer? This solution may not be quite so neat, but is good enough, and saves the student the task of learning a new method.

The author is constantly and terribly mixed up in his statements and notations, though of course not in his ideas, about number and magnitude. Unless the reader can supply a great deal he cannot properly interpret the statements on pages 9, 43, 66, 343, and I fear that there are exercises (for the freshman) on this last page which I could not myself answer with any assurance of agreeing with the author.

Slichter's Elementary Mathematical Analysis should be widely tried out, if only for the rest that it will give the teacher from the familiar beaten paths; there is a charming freshness about the work and, whether we like it or not, it is bound to be ranked as a distinct contribution to the theory and practice of freshman instruction in mathematics.

E. B. WILSON.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

PROBLEMS FOR SOLUTION.

ALGEBRA.

When this issue was made up, no solutions had been received for numbers 417–426.

426. Proposed by HERBERT N. CARLETON, West Newbury, Mass.

Find all the solutions of the equation $x\sqrt[x]{x} = x^x$.

427. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

If $r \sin(\theta + \alpha) = m$ and $r \cos(\theta + \beta) = n$, show that

$$r = \frac{\sqrt{m^2 + n^2 - 2mn \sin(\alpha - \beta)}}{\cos(\alpha - \beta)},$$

GEOMETRY.

When this issue was made up, no solutions had been received for numbers 447–8, 450–454.

455. Proposed by R. P. BAKER, University of Iowa.

Find the minimum triangle of assigned angles inscribed in a given triangle.

456. Proposed by J. W. CLAWSON, Ursinus College.

The interior and exterior bisectors of the angles A, B, C of a triangle meet the opposite sides in U, U'; V, V'; W, W' respectively. Circles are drawn on UU', VV', WW' as diameters (Circles of Apollonius.) Prove that (1) These three circles have a common chord. (2) The centre of the circumcircle lies on this common chord.

CALCULUS.

When this issue was made up, no solutions had been received for numbers 358, 361–2, 364–372, 374–5, and 377.

376. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that

$$\frac{1}{z} - \frac{1}{z}(1 - 2xz + z^2)^{\frac{1}{2}} = x + \frac{z}{2}\left(\frac{x^2 - 1}{1 - xz}\right) + \sum_{n=2}^{\infty} \frac{1, 3, 5 \cdots 2n - 3}{2, 4, 6 \cdots 2n} (x^2 - 1)^n \left(\frac{z}{1 - xz}\right)^{2n-1}$$

377. Proposed by W. D. CAIRNS, Oberlin College.

It is required to find a curve of the form $y = x(x - a)(x - b)$ such that the abscissas of the maximum and minimum values, as well as a and b, shall be positive integers.

MECHANICS.

When this issue was made up, no solutions had been received for numbers 289, 292–3, 295–299, 301, and 303.

302. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

A ball is projected from a given point at a given inclination β towards a vertical wall; determine the velocity so that after striking the wall the ball may return to the point of projection.

NUMBER THEORY.

When this issue was made up, no solutions bad been received for numbers 215–16, 218, 220, and 223–226.

226. Proposed by ELBERT H. CLARKE, Purdue University.

If 0! is taken equal to 1, and if k is any positive integer greater than or equal to 2, show that

$$\sum_{n=0}^{\infty} \frac{n!}{(k+n)!} = \frac{1}{(k-1)} \cdot \frac{1}{(k-1)!}.$$

227. Proposed by R. P. BAKER, University of Iowa.

Show that every rational number can be expressed as a finite sum $\sum_{n=m}^{n=m+k} \frac{a_n}{n}$, where a_n is either 0 or 1 and m is any positive integer.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

409. Proposed by C. E. GITHENS, Wheeling, W. Va.

Find integral values for the edges of a rectangular parallelopiped so that its diagonal shall be rational.

II. SOLUTION BY ARTEMAS MARTIN, Washington, D. C.

On pp. 269–273 of the October, 1914, MONTHLY, W. C. Eells has solved a different problem from the one proposed. In making $x^2 + y^2 = \square$, he adds another condition not required.

Let x, y, z be the edges and d the diagonal of the parallelepiped; then we have to satisfy the equation

$$x^2 + y^2 + z^2 = d^2.$$

It is not necessary that $x^2 + y^2$ be a square. Let us assume $x = a$, $y = b$, $z + c = d$, and we have

$$a^2 + b^2 + z^2 = (z + c)^2 = z^2 + 2cz + c^2,$$

which immediately gives

$$z = \frac{a^2 + b^2 - c^2}{2c} \qquad \text{and} \qquad d = \frac{a^2 + b^2 + c^2}{2c}.$$

Therefore

$$a^2 + b^2 + \left(\frac{a^2 + b^2 - c^2}{2c}\right)^2 = \left(\frac{a^2 + b^2 + c^2}{2c}\right)^2,$$

whatever be the values of a, b, c.

1. Taking $a = 1$, $b = 2$, $c = 1$, we find

$$1^2 + 2^2 + 2^2 = 3^2,$$

which is the *smallest* rational parallelepiped.

2. Taking $a = 2$, $b = 3$, $c = 1$, we have

$$2^2 + 3^2 + 6^2 = 7^2,$$

which is the smallest rational parallelepiped having its edges all different.

3. Taking $a = 1$, $b = 4$, $c = 1$, we get

$$1^2 + 4^2 + 8^2 = 9^2.$$

4. Let $a = 2$, $b = 6$, $c = 2$, and we have

$$2^2 + 6^2 + 9^2 = 11^2.$$

5. If $a = 3$, $b = 4$, $c = 1$, we get

$$3^2 + 4^2 + 12^2 = 13^2,$$

which, on p. 269, is stated to be "the smallest rational parallelopiped."

6. If we take $a = 8$, $b = 9$, $c = 5$, we will get

$$8^2 + 9^2 + 12^2 = 17^2.$$

7. Taking $a = 4$, $b = 8$, $c = 2$, we have

$$4^2 + 8^2 + 19^2 = 21^2.$$

8. If $a = 3$, $b = 4$, $c = 3$, then we get

$$12^2 + 15^2 + 16^2 = 25^2.$$

And so on, there being an infinite number of parallelopipeds whose edges and solid diagonals are rational integers.

The condition $x^2 + y^2 + z^2 = \square$ can be satisfied in many ways.—See *Mathematical Magazine*, Vol. II., No. 5 (October, 1891), pp. 71–74. Other methods are given in a forthcoming paper which will appear in the third part of No. 12, Vol. II., of the *Mathematical Magazine.*

416A. Proposed by H. O. HANSON, East Elmhurst, N. Y.

Find the nth term and the sum of n terms of the series obeying the relation $u_i = u_{i-1} + 2u_{i-2}$ in terms of n and the first two terms, u_1 and u_2, these two terms being arbitrary.

This problem was incorrectly numbered 416.

SOLUTION BY S. A. JOFFE, New York City.

Adding u_{i-1} to both members of the given relation, we obtain the following recurring formula:

$$u_i + u_{i-1} = 2(u_{i-1} + u_{i-2}). \tag{1}$$

If we replace i successively by n, $n-1$, $n-2$, $\cdots$, 3 and multiply the resulting $n-2$ equations, we have, after cancellation:

$$u_n + u_{n-1} = 2^{n-2}(u_2 + u_1). \tag{2}$$

Similarly

$$u_{n-2} + u_{n-3} = 2^{n-4}(u_2 + u_1),$$

$$u_{n-4} + u_{n-5} = 2^{n-6}(u_2 + u_1),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot,$$

the last equation of this type being $u_2 + u_1 = u_2 + u_1$, or $u_3 + u_2 = 2(u_2 + u_1)$, according as n is even or odd.

Adding all these equations, we have:

for $n =$ even, $\sum_{i=1}^{n} u_i = (2^{n-2} + 2^{n-4} + 2^{n-6} + \cdots + 1)(u_2 + u_1)$,

or

$$\sum_{i=1}^{n} u_i = \tfrac{1}{3}(2^n - 1)(u_2 + u_1); \tag{3}$$

and for $n =$ odd,

$$\sum_{i=1}^{n} u_i = (2^{n-2} + 2^{n-4} + 2^{n-6} + \cdots + 2)(u_2 + u_1) + u_1 = \tfrac{2}{3}(2^{n-1} - 1)(u_2 + u_1) + u_1,$$

or

$$\sum_{i=1}^{n} u_i = \tfrac{1}{3}(2^n - 2)(u_2 + u_1) + u_1. \tag{4}$$

Formulæ (3) and (4) express the sum of the first n terms, as required, in terms of n and the first two terms; the nth term u_n is found by subtracting $\sum_{i=1}^{n-1} u_i$ from $\sum_{i=1}^{n} u_i$, the result being for $n =$ even,

$$u_n = \tfrac{1}{3}(2^n - 1)(u_2 + u_1) - [\tfrac{1}{3}(2^{n-1} - 2)(u_2 + u_1) + u_1]$$
$$= \tfrac{1}{3}(2^{n-1} + 1)(u_2 + u_1) - u_1;$$

and for $n =$ odd, $u_n = [\tfrac{1}{3}(2^n - 2)(u_2 + u_1) + u_1] - \tfrac{1}{3}(2^{n-1} - 1)(u_2 + u_1)$

$$= \tfrac{1}{3}(2^{n-1} - 1)(u_2 + u_1) + u_1.$$

The last two formulæ may be combined into one, for n in general,

$$u_n = \tfrac{1}{3}[2^{n-1} + (-1)^n](u_2 + u_1) + (-1)^{n-1}u_1. \tag{5}$$

Also solved by A. M. HARDING, A. H. HOLMES, HORACE OLSON, and the PROPOSER.

417A. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Solve $E(x^2) - E(3x) = 7$, where $E(m)$ is the largest integer in m. The value of x is to be in the form $4 + (y/32)$ where y is an integer.

This problem was incorrectly numbered 417.

SOLUTION BY A. M. HARDING, University of Arkansas.

If we draw the graphs of the functions $E(x^2)$ and $E(3x) + 7$ we find

$$E(x^2) = E(3x) + 7 = 20, \quad \text{if} \quad \sqrt{20} < x < \sqrt{21}; \quad \text{i. e.} \quad 4.4721 < x < 4.5826;$$

$$E(x^2) = E(3x) + 7 = 21, \quad \text{if} \quad 14/3 < x < \sqrt{22}; \quad \text{i. e.} \quad 4.6667 < x < 4.6904;$$

$$E(x^2) = E(3x) + 7 = 2, \quad \text{if} \quad -5/3 < x < -\sqrt{2},$$

$$\text{i. e.} \quad -1.6667 < x < -1.4142.$$

Now we must have $x = 4 + y/32$ where y is an integer. Hence

$$4.4721 < 4 + y/32 < 4.5826, \qquad \therefore y = 16, 17, \text{ or } 18.$$

And

$$4.6667 < 4 + y/32 < 4.6904, \qquad \therefore y = 22.$$

$$-1.6667 < 4 + y/32 < -1.4142, \qquad \therefore y = -181, -180, \cdots, -174.$$

GEOMETRY.

430. Proposed by DANIEL KRETH, Wellman, Iowa.

The distance between A and B is always a feet. A travels along a straight path at the rate of v_1 miles per hour, and B starts at the same time in the path behind A and travels in a curve at the rate of v_2 miles per hour. How far will B travel to reach the path in front of A, and how far to reach the path again behind A?

SOLUTION BY ELIJAH SWIFT, University of Vermont.

Take the origin at the place where B starts and the Y-axis through A. Then the coördinates of A at any time, t, are $(0, a + v_1t)$, and those of B must satisfy the equations

$$(1) \quad x^2 + (y - a - v_1t)^2 = a,^2 \qquad (2) \quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = v_2^2.$$

Solving (1) for y, we have $y = -\sqrt{a^2 - x^2} + a + v_1t$, where the negative sign holds until B is abreast of A.

Substituting this value in (2), we have a quadratic equation in dx/dt, from which we find

$$\frac{dx}{dt} = \frac{\sqrt{a^2 - x^2}}{a^2}\{-v_1x + \sqrt{v_1^2x^2 - a^2v_1^2 + a^2v_2^2}\},$$

the plus sign of the second radical being taken since dx/dt is positive.

From equation (2), we deduce $ds = v_2dt$. Hence, we have

$$s \equiv v_2t = 2 \cdot v_2 \cdot a^2 \int_0^a \frac{dx}{\sqrt{a^2 - x^2}\{-v_1x + \sqrt{v_1^2x^2 - a^2v_1^2 + a^2v_2^2}\}},$$

as the total distance is evidently twice as far as the distance B must cover to be abreast of A. This would seem to be an elliptic integral and hence not easily evaluated.

The same reasoning would hold on the return, except for signs, and we obtain the same expression for s except for the sign of v_1x, which would be plus.

Note.—This problem should evidently have been listed under Calculus.

432. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Having given a tetrahedron, a, b, c, d, e, f, find an expression for the radius of the sphere which is tangent to the six edges.

Solution by J. W. Clawson, Collegeville, Pa.

First, a solution is not possible in the general case. Eight spheres can be drawn to touch any four of the six edges, but none of them will touch the other two edges unless certain conditions are satisfied.

If the tetrahedron is of such a shape that a sphere can be *inscribed* to touch all the edges, the planes of the four faces of the tetrahedron cut the sphere in circles inscribed to the triangles which form the faces. At each point where one of these four circles touches an edge, another of the circles also touches it. Now if T is the point where the in-circle of the triangle ABC, whose edges are a, b, c touches the side AC, then $CT = (a + b - c)/2$; and if T is the point where the in-circle of the triangle ACD whose edges are f, d, b touches the side AC, then $CT = (b + f - d)/2$.

Therefore $a + d = c + f$.

Similarly it can be shown that $a + d = b + e$.

Hence, a sphere can be drawn to touch the six edges internally if

$$a + d = b + e = c + f \tag{1}$$

Similarly it can be shown that a sphere can be drawn tangent to the six edges, touching b, c, d *produced*, if $a - d = e - b = f - c$; that a sphere can be drawn tangent to the six edges, touching a, b, c *produced* if $a - d = e - b = c - f$; one touching a, b, f, *produced* if $a - d = b - e = f - c$; one touching e, d, f *produced* if $a - d = b - e = c - f$.

Thus, if either

$$a + d = b + e = c + f,$$

$$\text{or}\quad a - d = b - e = c - f, \qquad \text{or}\quad a - d = b - e = f - c,$$

$$\text{or}\quad a - d = e - b = c - f, \qquad \text{or}\quad a - d = e - b = f - c,$$

a sphere can be found to touch the six edges.

If any face is an equilateral triangle and the other three edges are equal, two such spheres can be drawn, one inscribed, the other escribed, touching the sides of the equilateral triangle and the three other edges produced.

Using this fact, a little more reduction leads to the result

$$R = \frac{(a+e-f)}{24V}\sqrt{[(e-c)^2-d^2][(c-a)^2-b^2][f^2-(a-e)^2]}, \tag{3}$$

where

$$\begin{aligned}144V^2 = {} & a^2d^2(e^2+f^2-a^2)+b^2e^2\,(f^2+a^2-e^2)+c^2f^2(a^2+e^2-f^2)\\ & -a^2(d^2-b^2)(d^2-c^2)-e^2(b^2-c^2)(b^2-d^2)-f^2(c^2-d^2)(c^2-b^2)-a^2e^2f^2.\end{aligned} \tag{4}$$

Seven similar expressions can be found for the radii of spheres tangent to the four edges a, c, e, f, touching one or more of the edges produced beyond the tetrahedron.

Thirdly, applying the conditions (1) to (3), we can obtain the required answer in a symmetrical form. Now (3) is the radius of a sphere touching the four edges a, c, e, f. If this sphere touches also the edges b, d, that is, if $a+d=b+e=c+f$, then

$$R = \frac{1}{24V}\sqrt[3]{\begin{aligned}&(a+b-c)(a-b+c)(-a+b+c)(b+d-f)\\&(b-d+f)(-b+d+f)(c+d-e)(c-d+e)\\&(-c+d+e)(a+e-f)(a-e+f)(-a+e+f),\end{aligned}} \tag{5}$$

where V has the value given in (4), which is already expressed in a symmetrical form. This is the required solution.

It may be added that, if A_1, A_2, A_3, A_4 are the areas of the four face-triangles, and r_1, r_2, r_3, r_4 the radii of the circles inscribed to these triangles, the above expression takes the simple form

$$R = \frac{2}{3V}\sqrt[3]{A_1A_2A_3A_4r_1r_2r_3r_4}. \tag{6}$$

If the tetrahedron is regular the expression reduces to

$$R = \frac{a}{2\sqrt{2}}.$$

Similarly the radii of the four spheres R_1, R_2, R_3, R_4 which touch three edges and the other three edges produced, are

$$R_1 = \frac{2}{3V}\sqrt[3]{A_1A_2A_3A_4r_1r_{21}r_{31}r_{41}}; \qquad R_2 = \frac{2}{3V}\sqrt[3]{A_1A_2A_3A_4r_2r_{12}r_{32}r_{42}};$$

$$R_3 = \frac{2}{3V}\sqrt[3]{A_1A_2A_3A_4r_3r_{13}r_{23}r_{43}}; \qquad R_4 = \frac{2}{3V}\sqrt[3]{A_1A_2A_3A_4r_4r_{14}r_{24}r_{34}},$$

where r_{pq} is the radius of the circle escribed to the triangle A_p which touches the edge common to A_p and A_q and the other two edges of A_p produced.

If the tetrahedron is regular, $R_1 = R_2 = R_3 = R_4 = \dfrac{3a}{2\sqrt{2}}$.

CALCULUS.

358. Proposed by C. N. SCHMALL, New York City.

About a given circle circumscribe the smallest parabola.

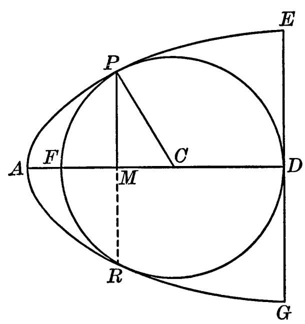
SOLUTION BY HORACE OLSON, Chicago, Illinois.

We assume that by the smallest parabola is meant the smallest segment of a parabola having its bounding ordinate tangent to the given circle.

Let $PFRD$ be the given circle whose center C is the origin of rectangular coördinates. Then, without loss of generality, the parabola may be assumed to have its axis on the axis of abscissas AD.

Let EAG be the required parabola. Then, if r is the radius of the circle, we have $x^2 + y^2 = r^2$ for the circle, and $y^2 = 2p(x + k)$ for the parabola, where p is the distance from the focus to the directrix and k is the distance AC.

Since P is the point of tangency, solving these two simultaneous equations subject to the condition of tangency, we have $r^2 - 2kp + p^2 = 0$.



Hence, $k = p/2 + r^2/2p$.

Then

$$AD = AC + CD = \frac{p}{2} + \frac{r^2}{2p} + r = \frac{(r+p)^2}{2p},$$

and

$$ED = \sqrt{2p(AD)} = (r + p).$$

Area of $EAG = a = \frac{2}{3}AD \cdot EG = \frac{4}{3}(r+p)^2/2p \cdot (r+p) = \frac{2}{3}(r+p)^3/p$.
Equating to zero the derivative of a with respect p, we have

$$\frac{da}{dp} = \frac{2}{3} \cdot \frac{3p(r+p)^2 - (r+p)^3}{p^2} = 0;$$

whence

$$(p + r)^2 = 0 \quad \text{and} \quad 2p - r = 0.$$

Hence, $p = -r$ or $p = \frac{1}{2}r$. The value $p = -r$ gives neither a maximum nor a minimum. The value $p = \frac{1}{2}r$ gives a minimum, and the equation of the corresponding parabola is $y^2 = r(x + 5r)/4$.

A similar solution was received from the PROPOSER.

359. Proposed by W. D. CAIRNS, Oberlin College.

Examine for maxima and minima

$$f(x) = e^{-cx}(1 + \cos x) \quad (c > 0)$$

SOLUTION BY A. M. HARDING, University of Arkansas.

$$f'(x) = -e^{-cx}(c + c\cos x + \sin x).$$

$$f''(x) = e^{-cx}(c^2 + c^2\cos x + 2c\sin x - \cos x).$$

Now $f'(x) = 0$ only when $c + c\cos x + \sin x = 0$, that is, when

$$c(1 + \cos x) + \sin x = 0,$$

or

$$2c\cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2} = 0.$$

Hence,

$$\cos\frac{x}{2} = 0, \quad \text{and} \quad x = \pi, 3\pi, 5\pi, \cdots (2n-1)\pi, \cdots$$

or

$$c\cos\frac{x}{2} + \sin\frac{x}{2} = 0, \quad \text{and} \quad x = 2\arctan(-c).$$

When $x = \pi, 3\pi, 5\pi, \cdots, (2n-1)\pi, \cdots, f''(x) = -c(2n-1)\pi$, which is always positive.

Hence, the minimum value of $f(x)$ is obtained by giving x any of these values. When $x = 2\arctan(-c)$, that is, $c\cos x/2 + \sin x/2 = 0$,

$$\begin{aligned} f''(x) &= e^{-cx}[\{c^2(1+\cos x) + c\sin x\} + c\sin x - \cos x] \\ &= e^{-cx}\left[2c\cos\frac{x}{2}\left(c\cos\frac{x}{2} + \sin\frac{x}{2}\right) + 2c\sin\frac{x}{2}\cos\frac{x}{2} - \cos x\right] \\ &= e^{-cx}\left[2c\sin\frac{x}{2}\cos\frac{x}{2} - \cos x\right] = e^{-cx}\left[2\sin\frac{x}{2}\left(-\sin\frac{x}{2}\right) - \cos x\right] \\ &= -e^{-cx}\left[2\sin^2\frac{x}{2} + \cos x\right] = -e^{-cx}[1 - \cos x + \cos x] = -e^{-cx}. \end{aligned}$$

Now $-e^{-cx}$ is negative for the above value of x. Hence the maximum value of $f(x)$ is obtained by giving x the value $2\arctan(-c)$.

Also solved by W. C. EELLS, PAUL CAPRON, H. C. FEEMSTER, G. W. HARTWELL and the PROPOSER.

MECHANICS.

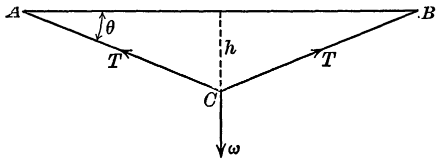
286. Proposed by C. N. SCHMALL, New York City.

A slightly elastic string is just long enough to reach between two hooks on the same horizontal line. A ring of weight w is placed at its middle point. Show that the ring will sink through a distance $h = a\sqrt[3]{3e\omega/2}$, where e is the elasticity of the string and $2a$ the distance between the two hooks.

SOLUTION BY B. F. FINKEL, Drury College.

Since w is at the middle point of the string, the tension T in the two halves of the string is the same when the string is in equilibrium. Let θ be the angle which the string makes with the horizontal line.

Then $AC = a \sec \theta$. Hence the strain in AC is



$$\frac{a \sec \theta - a}{a} = \sec \theta - 1.$$

By Hooke's Law, we have $\quad T/(\sec \theta - 1) = e. \qquad (1)$
For equilibrium, we have $\quad 2T \sin \theta = w. \qquad (2)$
Hence,

$$2e \sin \theta(\sec \theta - 1) = w,$$

or

$$\tan \theta - \sin \theta = \frac{w}{2e}. \qquad (3)$$

Since the string is only slightly elastic, θ must be small.

When θ is small we may take $\tan \theta = \theta + (\theta^3/3)$ and $\sin \theta = \theta - (\theta^3/6)$. Hence, substituting these values in (3) and solving for θ, we have $\theta = \sqrt[3]{w/e}$.

But $\tan \theta = h/a$, or $\theta + (\theta^3/3) = h/a$. Substituting for θ and solving for h, we have

$$h = a\left\{ \sqrt[3]{\frac{w}{e}} + \frac{w}{3e} \right\}.$$

The proposer's result is incorrect, as the test of physical dimensions shows.

290. Proposed by B. F. FINKEL, Drury College.

A fox, pursued by a hound, is running with uniform velocity over a frail arch in the form of a cycloid: the hound stops at a weak point of the arch, then tumbles through, and reaches the level ground with a velocity equal to that of the fox. Prove that the fox exerted no normal pressure on the arch at the point where the hound fell through. Walton's *Problems in Theoretical Mechanics*, p. 662.

SOLUTION BY THE PROPOSER.

Let $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ be the parametric equations of the cycloidal arch and v_0, the speed of the fox. Then at the point where the hound tumbled through, we have the relation $v_0^2 = 2gy_0$, where y_0 is the ordinate of the arch at that point.

Now the centripetal force of the fox is $m(v_0^2/R)$, where m is his mass and R the radius of curvature of the arch. The force exerted by gravity in opposition to this force is $mg \cos \phi$, where ϕ is the direction of the arch at any point.

Hence, the normal pressure on the arch at any point, assuming $m = 1$, is

$$P = \frac{v_0^2}{R} - g \cos \phi.$$

From the equations of the arch, we have

$$\frac{dy}{dx} = \tan \phi = \cot \frac{\theta}{2} = \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right).$$

Whence, $\phi = (\pi/2) - (\theta/2)$. Also

$$R = \frac{[1 + (dy/dx)^2]^{-\frac{3}{2}}}{d^2y/dx^2} = 4a \sin \theta/2$$

Hence, $P = v_0^2 \div (4a \sin \theta/2) - g \sin \theta/2$. But, since $y = \theta(1 - \cos \theta)$,

$$2 \sin^2 \frac{\theta}{2} = \frac{y}{a}.$$

Hence,

$$P = \frac{v_0^2 - 4ag \sin^2 \theta/2}{4a \sin \theta/2} = \frac{v_0^2 - 2gy}{4a \sin \theta/2},$$

But, at the point where the hound tumbled through, $y = y_0 = (v_0/2g)^2$. Hence, at that point, $P = 0$.

NUMBER THEORY.

212. Proposed by C. N. SCHMALL, New York City.

Given any positive integer N greater than 1; to prove that the sum of all the positive integers less than N and prime to N equals $\frac{1}{2}N \cdot \phi(N)$.

SOLUTION BY H. C. FEEMSTER, York College, Nebraska.

Let $N = a^h b^k c^l \cdots$, where $a, b, c, \cdots$ are primes. Then

$$\phi(N) = N \cdot \frac{a-1}{a} \cdot \frac{b-1}{b} \cdot \frac{c-1}{c} \cdot \cdots = N\left[1 - \Sigma\frac{1}{a} + \Sigma\frac{1}{ab} - \Sigma\frac{1}{abc} + \cdots\right],$$

the number of positive integers less than N and prime to N.

For every number $p < N/2$ and prime to N, there is a number $N - p > N/2$ and less than N and prime to N. Now there are $\phi(N)$ of these numbers or $\frac{1}{2}\phi(N)$ pairs of these numbers. But the sum of each pair is N, so the entire sum is $\frac{1}{2}N \cdot \phi(N)$, as required.

213. Proposed by R. D. CARMICHAEL, Indiana University.

Prove that no relatively prime integers x and y exist such that the difference of their fourth powers is a perfect cube.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

We are to prove the non-existence of any equation $x^4 - y^4 = z^3$. There are two cases to consider: (1) *x and y both odd*; (2) *one even and one odd.*

In the second case $x^2 + y^2$ and $x^2 - y^2$ are prime to each other, since any common factor would divide their sum $2x^2$, and difference $2y^2$; but these have, by hypothesis, 2 as their only common factor, and 2 is not a factor of $x^2 + y^2$ because x is even and y odd, or vice versa.

Since the product of $x^2 + y^2$ and $x^2 - y^2$ is a perfect cube, each of the factors must be, say, $x^2 + y^2 = a^3$, $x^2 - y^2 = b^3$. Call $2xy = c$. Then $a^6 - b^6 = c^2$, which is impossible. (See Number Theory, problem 209 in the March, 1914, MONTHLY which denies the existence of such an equation.)

Suppose now that x and y are both odd. Let $x = 4k \pm 1, y = 4l \pm 1$. Then $x^2 + y^2 = 16(k^2 + l^2) + 8(\pm k \pm l) + 2$, which is divisible by 2 but not by 4; and $x^2 - y^2 = 16(k^2 - l^2) + 8(\pm k \pm l)$, which is divisible by 8. Since $z^3 = (x^2 + y^2)(x^2 - y^2)$, it is divisible by 16 and hence by 64; and $x^2 - y^2$ by 32. Accordingly, we may write $x^2 + y^2 = 2\alpha^3$, $x^2 - y^2 = 32\beta^3$. We now see that if $x = 4k + 1$, $y = 4l + 1$, then $x + y$ is divisible by 2, but not by 4; any other combination of signs leads to this result for $x + y$ or $x - y$. We may then write, in the one case, $x + y = 2\gamma^3$, $x - y = 16\delta^3$. The other case may be similarly treated. Then

$$(x + y)^2 + (x - y)^2 = 4\gamma^6 + 256\delta^6 = 2x^2 + 2y^2 = 4\alpha^3.$$

Hence $\gamma^6 + 64\delta^6 = \alpha^3$, or $(\gamma^2)^3 + (2\delta^2)^3 = \alpha^3$, which is impossible. Hence there are no integral values of x and y *prime to each other* which satisfy the given equation.

Also solved by J. L. Riley

QUESTIONS AND DISCUSSIONS.

Edited by U. G. Mitchell.

At the time of making up copy for this issue replies had not yet been received for numbers 4, 8, 11, 12, 13, 16, 18.

NEW QUESTIONS.

22. What can the Colleges do toward improving the teaching of mathematics in secondary schools?

23. What should be done with the theory of limits in elementary geometry? Should the recommendation of the National Committee of Fifteen on Geometry Syllabus be universally adopted? If not, what better disposition of the subject can be made?

REPLIES.

10. What use has been made of regular conference periods for assistance to individual students of secondary and college mathematics, and what services may they render?

Reply by Clyde S. Atchison, Washington and Jefferson College.

For the past two years, one hour each day has been set aside for conference in mathematics, during which time one member of the department is in his class room ready to assist students in settling their individual difficulties. The privilege thus offered has been much appreciated by a majority of the students, of whom many seize the opportunity for extra instruction to enlarge their grasp of the subject, while others, of less ability or unfortunate preparatory training, make use of the occasion to have explained to them those points to which only a limited amount of time can be devoted in the regular class work. A frank announcement in the class room, that no time will be wasted with a man who does not keep up with the work of the course, and that a conference hour is not a time

when a man's lessons will be prepared for him, has forestalled any attempt on the part of students to take an unfair advantage of the hour.

The personal touch with the men, in a way which is not possible during the class period, together with what has been accomplished in the way of developing both the weaker students and those of more than average ability, has made these conference hours of inestimable value.

3. In connection with the theory of the conduction of electricity through gases, one is led to the differential equation

$$y\frac{d^2y}{dx^2} + a\left(\frac{dy}{dx}\right)^2 + b\frac{dy}{dx} + cy + d = 0, \tag{1}$$

where a, b, c, d are constants. For unrestricted values of a, b, c, d the solution of this differential equation presents peculiar difficulties, the series solutions obtained by the customary methods having (apparently) too small a range of convergence to be satisfactory from the point of view of electrical theory. The general solution of this equation is wanted in case it can be found. If no general solution is obtained for unrestricted a, b, c, d, it is desirable to know special values of a, b, c, d or special relations among a, b, c, d which make it possible to find the general solution; and this solution is desired in each case.

REPLY BY BARNEM LIBBY, University of Michigan.

The given equation may be written

$$yy'' + ay'^2 + by' + cy + d = 0. \tag{1}$$

Consider the equation

$$yy'' + ay'^2 + by' + cy = 0. \tag{2}$$

This is equivalent to

$$[ayy' + by]' + y[(1 - a)y' + cx]' = 0. \tag{3}$$

Hence (3) is exact if the coefficient of y is zero; and this is the case if $c = 0$ and $a = 1$.

We then get in (1)

$$[yy' + by]' = -d, \tag{4}$$

whence,

$$yy' + by = -d \cdot x + l, \tag{5}$$

l being a constant of integration.

To integrate

$$ydy + (by + d \cdot x - l)dx = 0, \tag{5'}$$

we put

$$\begin{cases} x = x_1 + \alpha, \\ y = y_1 + \beta. \end{cases}$$

Then $dx = dx_1$, $dy = dy_1$ and (5′) becomes

$$(y_1 + \beta)dy_1 + (by_1 + d \cdot x_1 + b\beta + d \cdot \alpha - l)dx_1 = 0. \tag{6}$$

If now $\beta = 0$ and $d \cdot \alpha = l$, this becomes

$$y_1dy_1 + (by_1 + d \cdot x_1)dx_1 = 0. \tag{6'}$$

Let $y_1 = vx_1$, then $dy_1 = vdx_1 + x_1dv$, and then

$$vx_1(vdx_1 + x_1dv) + (bvx_1 + d \cdot x_1)dx_1 = 0,$$

or

$$v(vdx_1 + x_1dv) + (bv + d)dx_1 = 0, \tag{7}$$

i. e.,

$$(v^2 + bv + d)dx_1 + vx_1dv = 0,$$

or

$$\frac{vdv}{v^2 + bv + d} = -\frac{dx_1}{x_1}, \tag{7'}$$

or

$$\frac{\frac{1}{2}(2v + b) - \frac{1}{2}b}{v^2 + bv + d} = -\frac{dx_1}{x_1}.$$

Hence,

$$\log (v^2 + bx + d)x_1^2 - b\int \frac{dv}{v^2 + bv + d} = k. \tag{8}$$

Evaluating the indefinite integral and substituting the values of v^2 and v, we obtain a solution for y in terms of x.

NOTES AND NEWS.

EDITED BY W. D. CAIRNS.

Mr. JOHN BRANDEBERRY, A.B., Mt. Union College, 1914, and Miss MARVEL C. HORN, A.B., Ohio State University, 1914, have been elected graduate assistants in mathematics at Ohio State University, for the present academic year.

Mr. RAYMOND DU HADWAY, who has been studying at Göttingen the past year was forced to leave Germany on account of the war. He has taken a place to teach mathematics in Washington University, St. Louis, Mo.

Professors PIERRE BOUTROUX and J. H. M. WEDDERBURN, of the department of mathematics, Princeton University, have been granted leaves of absence; the former is enrolled in the French service, the latter in the British service.

At the University of Texas, Dr. DAVID F. BARROW has been appointed instructor in applied mathematics, and Mr. F. A. LA MOTTE instructor in pure mathematics. The new courses being given are: The calculus of variations, by Mr. ETTLINGER; and the mathematics of investment and life insurance, by Professor DODD.

It was proposed to hold the next meeting of the International Commission on the Teaching of Mathematics at Munich, Germany, August 2–5, 1915; but this meeting has been indefinitely postponed on account of the war. The main sub-

ject for discussion was to have been the preparation, theoretic and practical, of teachers of mathematics for the various grades. At the request of the Central Committee Professor Gino Loria, Genoa, Italy, has assumed general charge of the work relative to teachers of secondary mathematics.

According to the list of members published in the July, 1914, issue of the *Revista de la Sociedad Matemática Española,* the number of members of the Spanish Mathematical Society is now 436. Three of these members live in the United States.

The October, 1914, number of the *Journal of the Indian Mathematical Society* contains an article by PHILIP E. B. JOURDAIN entitled "The Theories of Irrational Numbers, Part I." According to the author's own words "the aim of this historical and critical study is somewhat different from that of the other works known to us which deal with the development of the theory of convergence and allied topics. We shall, in fact, be concerned primarily with questions of principle."

The National Academy of Sciences began in January, 1914, the publication of a monthly periodical called *Proceedings.* Professor E. H. MOORE, of the University of Chicago, is the mathematical member on the editorial staff. Professor E. B. WILSON, Massachusetts Institute of Technology, is the managing editor. The articles are expected to be short and to exhibit a summary of the most important results obtained by Americans in various fields of science.

Professor E. W. HOBSON gave six lectures at Cambridge University during the Easter term of last year on the history of the problem of the quadrature of the circle. These lectures have been published by the Cambridge University Press in a book entitled "Squaring the Circle. A History of the Problem." The price of the book is three shillings.

A London press dispatch says that "King George has approved the presentation by the council of The Royal Society of a royal gold medal to Professor ERNEST W. BROWN of Yale University for his investigations in astronomy."

WILLIAM FROTHINGHAM BRADBURY, author of textbooks on algebra, geometry, and trigonometry, died in Boston on October 22, 1914, at the age of 86 years. From 1886 to 1910 he was headmaster of the Boston Latin School.

MICHAEL A. MCGINNIS, who was the author of a book in which he claimed (falsely, of course) to have devised a universal solution for both numerical and literal equations, died recently in Kansas City. It is claimed by many who knew him intimately that, but for strong drink and dishonest business methods, he might have made a noteworthy record as a mathematician.

J. M. Greenwood, who, for many years and to the time of his death late last autumn, was superintendent of schools in Kansas City, Mo., was an ardent friend of the Monthly, having been a regular subscriber for a long period. He was a lover of mathematics for its own sake, spending many of his spare hours not only in mathematical recreations but in genuine mathematical study. For years he had been chairman of the appropriations committee of the National Education Association, where he always showed a keen appreciation of all investigations which had to do with improvement of mathematical teaching, one of his liberal recommendations being for the appropriation of funds to finance the work of the National Committee of Fifteen on Geometry Syllabus.

Professor Giovan Battista Guccia, of the University of Palermo, died on October 29, 1914. He was the founder of the Circolo Matematico di Palermo and editor of its official publication the *Rendiconti*.

The Division of Mathematics of Harvard University announces that hereafter two appointments will be made each year to the Benjamin Peirce Instructorships in mathematics, which carry a stipend of $1,000 to $1,200 and allow the incumbents to pursue courses for higher degrees while giving instruction to the extent of about ten and one half hours per week. These instructors may be reappointed but not for more than three years.

"The Training of Mathematics Teachers" is the title of a paper read by Professor G. A. Miller at the 1914 meeting of the Central Association of Science and Mathematics Teachers. The paper appears in the January, 1915, issue of *School Science and Mathematics*. A review of this stimulating address, together with a report of other features of this meeting will appear in a later issue of the Monthly.

Miss Olive C. Hazlett, a graduate student at the University of Chicago, is the author of an article on "Invariantive characterization of some linear associative algebras" which appeared in the *Annals of Mathematics* for September, 1914.

Dr. Thomas E. Mason, who took his doctorate at Indiana University last year, is instructor in mathematics at Purdue University. His thesis, entitled "Character of the solution of certain functional equations," was published in the *American Journal of Mathematics* for October, 1914.

Professor G. A. Miller, of the University of Illinois, published an article in *The Popular Science Monthly* for November, 1914, which the readers of the American Mathematical Monthly will be glad to see. It is entitled "Recent mathematical activities." It is an excellent résumé of matters with which everyone interested in mathematics should be familiar.

"The Uses for Mathematics" is the title of an article in *Science* of November 13, 1914, by Professor S. G. Barton of Flower Observatory, University of Pennsyl-

vania, in which the great debt of the sciences to mathematics is dwelt upon. A list is given of 104 titles from the last edition of the *Encyclopedia Britannica* the treatment of which requires the infinitesimal calculus. The article closes with the following quotation from Sir John Herschel:

"Admission to its sanctuary (the sanctuary of astronomy) and to the privileges and feelings of a votary is only to be gained by one means—sound and sufficient knowledge of mathematics, the great instrument of all exact inquiry, without which no man can ever make such advances in this or in any other of the higher departments of science as can entitle him to form an independent opinion on any subject of discussion within their range."

The Association of Mathematics Teachers of New Jersey is a new organization which held its first meeting November 7, 1914, with an attendance of about sixty. While composed chiefly of high-school teachers the organization is not to be conducted, we understand, so largely along so-called "normal" lines as has been the case in many instances, but rather the *common* interests of high school and college mathematics are to be conserved and the programs are to contain more papers dealing with subject matter of at least as high a grade as the calculus. To this end, besides several papers of the usual type, two of the newer type were given at this initial meeting, namely: on "Number and the Quadratic" by Professor RICHARD MORRIS of Rutgers College, and on "The Mechanics of Aviation" by Professor L. P. EISENHART of Princeton University. These papers were reported as admirably well adapted to the end desired and the resulting effect seemed most satisfactory. This is a step along a line similar to that undertaken by the California teachers as reported in this issue of the MONTHLY. We commend the California reading course to the New Jersey association. A moving spirit in this organization is HARRISON E. WEBB, of the Central High School, Newark, N. J.

The winter meetings of the American Mathematical Society held in Chicago on December 28, 29, 1914, and in New York on January 1, 2, 1915, were among the most enthusiastic and largely attended in many years. At Chicago there were sixty-one members present and twenty-two papers read. At New York there were ninety-five members present and thirty-two papers were read. At each meeting a dinner was held on the evening of the first day and thus opportunity was offered for social intercourse which was enjoyed to the full. These meetings have grown to be most inspiring occasions and are well worth the effort and expense of even long journeys in order to attend them. The attendance in New York was the largest ever recorded at a meeting of the Society, and the ten or more members from the West who helped to swell this number were amply repaid in the coin of friendly greeting, extended acquaintance, and mutual fellowship. The next Summer meeting of the Society will be in San Francisco early in August.

THE AMERICAN MATHEMATICAL MONTHLY

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THE HISTORY OF ZENO'S ARGUMENTS ON MOTION.

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

IV.

By FLORIAN CAJORI, Colorado College.

4. EARLY DISCUSSIONS OF LIMITS: GREGORY ST. VINCENT, GALILEO, HOBBES.

Limits in the Fifteenth and Sixteenth Centuries. With the fifteenth century new mathematical ideas appear. These germs are found in Greek philosophy, but they failed to develop during the dark centuries. In the fifteenth century the German cardinal, Nicolaus Cusanus (1401–1465), considered variability without being able to apply it successfully; he advanced the notion of a limit, though unable to pass correctly to the limit; he entertained the notion of infinitesimals but was not able to use them in an infinitesimal calculus.[1] He held that rules developed for the finite lose their validity for the infinite—a statement which later thinkers have not always heeded sufficiently. A point moving with infinite velocity in a circle is each moment in every position on the circle; hence it is at rest.

During the century, or century and a half, after Cusanus, concepts of limits and processes involving the passing to the limit begin to appear in different parts of Europe, like flowers on a field in early spring. Perhaps first in time, in the development of ideas considered by Cusanus, is Giovanno B. Benedetti, a distinguished forerunner of Galileo, who brought out a publication in 1585 at Turin, Italy. As early as 1586, and again in 1608, Simon Stevin at Leyden exhibited the process of passing to the limit.[2] In 1604 the Italian mathematician, Luc Valerio, published at Rome a treatise, *De centrogravitatis*, which contains a remarkable approach to the modern idea of limits.[3] In Galileo's celebrated

[1] K. Lasswitz, *Geschichte der Atomistik*, Hamburg und Leipzig, 1. Bd., 1890, pp. 283, 284, 287, See also Max Simon, "Cusanus als Mathematiker," *Festschr. H. Weber*, Leipzig und Berlin, 1912. pp. 298–337.

[2] H. Bosmans, "Sur quelques exemples de la méthode des limites chez Simon Stevin," *Annales de la société scientifique de Bruxelles*, T. 37, 1912–13, 2. fascicule.

[3] H. Bosmans, "Les démonstrations par l'analyse infinitésimale chez Luc Valerio," *Annales de la Société scientifique de Bruxelles*, T. 37, 1912–13, 2. fascicule; C. R. Wallner, "Ueber die Entstehung des Grenzbegriffes," *Bibliotheca mathematica*, 3. F., Bd. IV, 1903, p. 250,

discourses on mechanics and falling bodies (1638) there are frequent instances of limits. In the Netherlands again, Gregory St. Vincent, whose researches have, until recently, hardly received the recognition they deserve, was familiar with the writings of Luc Valerio, and himself contributed toward laying the foundations for the infinitesimal calculus. Similar studies bearing on the concept of a limit are due to Andreas Tacquet of Antwerp, and to John Wallis in his *Arithmetica infinitorum*, 1655, who were both familiar with the *Opus geometricum* of Gregory St. Vincent.[1]

We proceed now to a special mention of discussions of Zeno's arguments. Benedetti, whom we mentioned above, held that the flying arrow, thought of at a point in its path, does not cover a finite distance, but it differs from an arrow at rest by possessing the attribute of velocity which persists even in an infinitesimal time and space.[2] Direct reference to Zeno in a manner which exhibits reckless following of the great dialectician is found in Giuseppe Biancani of Bologna who about 1615 sought to establish the incommensurability of two lines by the consideration that a supposed common measure could not be applied to either line, because the measure must first be applied to half of it, and before that to half of that half, and so on to infinity, which is as impossible an operation as Zeno's "Dichotomy."[3]

Speculations of Galileo. Far more successful than earlier writers in the application of infinitesimals were Kepler and Cavalieri, but more important to us at present are the speculations of Galileo. Galileo approached the problem of infinite aggregates with a keenness of vision and an originality which was not equalled before the time of Dedekind and Georg Cantor. Galileo's dialogues on mechanics, *Discorsi e Dimostrazioni matematiche*, 1638, opens the "first day" with a discussion of divisibility and continuity of matter and space.[4] Salviati, who in general represents the author's own ideas, says, [5]"the infinite is inconceivable to us, as is the last indivisible." Simplicio, who in these dialogues is the spokesman of Aristotelian scholastic philosophy, remarks that "the infinity of points on a longer line must be greater than the infinity of points on a shorter one." Then come the remarkable words of Salviati:

"These difficulties arise because we with our finite minds discuss the infinite, attributing to the latter properties derived from the finite and limited. This, however, is not justifiable; for the attributes great, small and equal are not applicable to the infinite, since one cannot speak of greater, smaller, or equal infinities. . . . If now I ask how many squares are there, one can answer with truth, just as many as there are roots; for every square has a root,[6] every root has a square, no square has more than one root, no root more than one square. . . . I see no escape, except to say: the totality of numbers is infinite, the totality of squares is infinite, the totality of roots is infinite; the multitude of squares is not less than the multitude of numbers, neither is

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[1] C. R. Wallner, *loc. cit.*, p. 257.

[2] K. Lasswitz, *op. cit.*, Vol. II, p. 17.

[3] J. C. Heilbronner, *Historia matheseos universæ*, Lipsiæ, 1742, p. 175.

[4] See a German translation in *Ostwald's Klassiker*, No. 11, pp. 24–37, also No. 24, p. 17; an English translation of the parts bearing on aggregates is given by E. Kasner in *Bulletin Am. Math. Soc.*, Vol. XI, 1904–5, pp. 499–501.

[5] *Ostwald's Klassiker*, No. 11, p. 29.

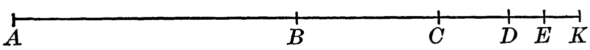
[6] Following the custom of his time, Galileo considers only one root of a positive number, namely the principal root.

the latter the greater; and, finally, the attributes equal, greater, and less are not applicable to infinite but solely to finite quantities."

We shall see that Galileo has been curiously misinterpreted by some writers, including Cauchy, as demonstrating here that an actual infinity has no existence. That there should be as many squares as there are integers altogether was taken as absurd; hence the existence of actual infinity was considered disproved. Galileo's skill in the use of the infinite in demonstrations is shown in the following passage on falling bodies:[1]

"If the velocity were proportional to the distance through which it has fallen or is to fall, then those distances would be passed over in equal times; thus, if the velocity with which a body overcomes four yards is to be double the velocity with which the first two yards were overcome, then the times needed for these two processes would be the same; but four yards can be overcome in the same time as two yards only in the case of instantaneous motion; we see on the contrary that the body needs time to fall, and that it needs less time for a fall of two yards than of four yards; hence it is not true that the velocities increase proportionally to the distance fallen."

Gregory St. Vincent. The most important discussion of Zeno given at this time is that by Gregory St. Vincent, in his *Opus geometricum quadraturæ circuli et sectionum coni,* published at Antwerp in 1647, but written apparently twenty-five years earlier. It is a massive volume of 1400 pages. Influenced in his geometrical researches by the medieval scholastic concept of the continuum, according to which a line divided repeatedly is not reduced to indivisible elements as taught by the atomists, but admits of being subdivided *ad infinitum,* Gregory St. Vincent took a step different from that of Archimedes. While, in his proofs, Archimedes kept on dividing, only until a certain degree of smallness was reached, St. Vincent permitted the subdivisions to continue *ad infinitum.* Using unlimited section in geometry he introduced a geometric series that was truly an *infinite* series.[2]



This much had been accomplished by at least one writer before him,[3] but, so far as now known, he is the first to apply the infinite geometric progression to the study of the "Achilles." Taking a definite line segment AK he divides it at B in a given ratio, then he divides BK in the same ratio at C, and so on. The segments $AB, BC, CD, \ldots$ form an infinite geometric progression. The points $C, D, E \ldots$ lie, all of them, between A and K; they approach K as near as we please, but (in accordance with scholastic philosophy) never reach it. As Gregory conceives this matter, K is an obstacle, so to speak, against the further advance of the series of points $A, B, C, \ldots,$ similar to a rigid wall. "Terminus progressionis est seriei finis, ad quem nulla progressio pertinget, *licet* in infinitum continuetur; sed quovis interuallo dato proprius ad eum *accedere* poterit." By "series" is meant the segment AK, by "progressio," the segments $AB, AC, \ldots.$

[1] *Ostwald's Klassiker,* No. 24, p. 17.

[2] Gregory St. Vincent, *Opus geometricum,* T. 1, pp. 51–56, 95–97; for our knowledge of this part of the book we are dependent entirely upon C. R. Wallner's account in the *Bibliotheca mathematica,* 3. F., Vol. IV, 1903, pp. 251–255.

[3] See H. Wieleitner in *Bibliotheca mathematica,* 3. F., Vol. 14, 1914, pp. 150–168.

Gregory states his conclusion thus: "Dico magnitudinem AK aequalem esse toti progressioni magnitudinum continue proportionalium, rationis AB ad BC in infinitum continuatae; siue quod idem est, rationis AB ad BC in infinitum continuatae terminum esse K." Considering the "Achilles" in this connection, he associates this paradox on motion for the first time definitely with the summation of an infinite series. Moreover, Gregory St. Vincent is the first writer known to us who states the exact time and place of overtaking the tortoise. So far as we are able to ascertain, Gregory was not troubled, in explaining the "Achilles," by the fact that in his theory, the variable does not *reach* its limit. Nor, apparently, did this matter trouble his readers. His mode of solving the problem appealed to many. We shall see that Leibniz makes special reference to it. Over a century after Gregory's publication, Saverien refers in his dictionary[1] to the "Achilles," "dont Gregoire de Saint Vincent a fait voir la fausseté." Formey gave Gregory St. Vincent's explanation in the article "Mouvement" in Diderot's *Encyclopêdie* (1754), later reprinted in the *Encyclopêdie mêthodique,* and in 1800 translated at Padova into the Italian language. The definition of a limit as given in the *Encyclopêdie mêthodique* does not allow the variable to surpass its limit but places no obstacle in the way of its reaching its limit.

Descartes, De Morgan and Others. Descartes at one time discussed the "Achilles." His treatment is much like that of Gregory St. Vincent. It is given in a letter of July, 1646, to Clerselier.[2] He lets Achilles, or in his place a horse, be, at the start, 10 leagues behind the tortoise, but moving ten times more rapidly than the latter. The real difficulty of the paradox he does not touch, for he says:

> "L'Achille de Zenon ne sera pas difficile à soudre, si on prend garde que, si à la dixiéme partie de quelque quantité on adioute la dixiéme de cette dixiéme, qui est une centiéme, & encore la dixiéme de cette derniere, qui n'est qu'une milliesme de la premiere, & ainsi à l'infini, toutes ces dixiémes jointes ensemble, quoy qu'elles soient suposées réellement infinies, ne composent toutesfois qu' une quantité finie, sçauvoir une neusiéme de la premiere quantité . . . Et la caption est en ce qu'on imagine que cette neusiéme partie d'une lieuë est une quantité infinie, à cause qu'on la divise par son imagination en des parties infinies."

Descartes looked upon the actually infinite as mysterious, but not impossible or absurd. He seemed to accept it in the abstract, but deny it in the concrete. At this time and even earlier (see the foregoing extracts from Galileo) there was talk about the finitude of the human mind and its consequent inability to conceive the infinite. This was ridiculed by De Morgan. He claimed that if the human mind is limited, we tacitly postulate the "unknowable"; moreover, even if the human mind were finite, there is no more reason against its conceiving the infinite than there is for a mind to be blue in order to conceive of a pair of blue eyes. Or, as De Morgan puts it in another place, the argument amounts to this, "who drives fat oxen should himself be fat." From Descartes to Hamilton,

[1] Saverien, *Dictionnaire universel de mathèmatique et de physique,* Paris, 1753, Art. "Mouvement."

[2] *Oeuvres de Descartes* par Charles Adam et Paul Tannery, T. IV, pp. 445–447.

says De Morgan,[1] this doctrine is accepted by many minds. But its genesis is found, as we have stated, long before Descartes.

A wholly different, but no more satisfactory explanation of "Achilles" comes from another Frenchman of that time, Pierre Gassendi, the physicist. In his view Zeno's proofs need no refutation, if with Epicurus one assumes not points but atoms. A difficulty seems to arise from differences in velocity of motion, for in the same time that a body moves over the physically indivisible, the more rapid body must travel over several indivisibles. In his opinion this difficulty may perhaps be overcome by conceiving motion as discontinuous, and slower motion as a mixture of rest and motion. To the senses motion would still seem continuous.[2] To those who experienced difficulties in accepting the existence of indivisible atoms, the capuchin, Casimir of Toulouse, offers an easy solution by reminding that angels had extension, yet were physically indivisible.[3]

It is worthy of note that John Dee, the famous astrologer who wrote an elaborate mathematical preface to Billingsley's edition of Euclid (1570), departs from the contention that two lines containing the same number of parts must be of equal length. He says:

"Our least Magnitudes can be divided into so many partes as the greatest. As, a Line of an inch long (with vs) may be divided into as many partes, as may the diameter of the whole world, from East to West: or any way extended."

Discussion of Thomas Hobbes. The earliest British writer, after Duns Scotus, to take up explicitly Zeno's arguments is the philosopher, Thomas Hobbes (1588–1679). In 1655 he wrote:[4]

". . . the force of that famous argument of Zeno against motion, consisted in this proposition, *whatsoever may be divided into parts, infinite in number, the same is infinite;* which he without doubt, thought to be true, yet nevertheless is false. For to be divided into infinite parts, is nothing else but to be divided into as many parts as any man will. But it is not necessary that a line should have parts infinite in number, or be infinite, because I can divide and subdivide it as often as I please; for how many parts soever I make, yet their number is finite; because he that says parts, simply, without adding how many, does not limit any number, but leaves it to the determination of the hearer, therefore we say commonly, a line may be divided infinitely; which cannot be true in any other sense."

With Hobbes, *infinite* is synonymous with *indefinite.* He takes an agnostic attitude toward problems of infinity:

"But when no more is said than this, *number is infinite,* it is to be understood as if it were said, this name *number* is an *indefinite* name. . . . And, therefore, that which is commonly said, that space and time may be divided infinitely, is not to be so understood, as if there might be any infinite or eternal division; but rather to be taken in this sense, *whatever is divided is divided into such parts as may again be divided.* . . . Who can commend him that demonstrates thus? 'If the world be eternal, then an infinite number of days, or other measures of time, preceded the birth of Abraham. But the birth of Abraham preceded the birth of Isaac; and therefore one

[1] A. De Morgan, "On Infinity; and on the Sign of Equality," in Trans. of the Cambridge Philosoph. Society, Vol. XI, p. 157, Cambridge, 1871 [read May 16, 1864].

[2] Gassendi, *Opera omnia*, 1658, I, p. 300a. An. I, p. 239; Lasswitz, *op. cit.*, Vol. II, p. 150.

[3] Lasswitz, *op. cit.*, Vol. II, p. 494.

[4] *The English Works of Thomas Hobbes*, Vol. I, London, 1839, pp. 63, 64, 413. Hobbes refers to Zeno's arguments also in his Latin works. See Thomae Hobbes, *Opera philosophica*, Vol. V, Londini, 1845, pp. 207–213.

infinite is greater than another infinite, or one eternal than another eternal; which,' he says, 'is absurd.' This demonstration is like his, who from this, that the number of even numbers is infinite, would conclude that there are as many even numbers as there are numbers simply, that is to say, the even numbers are as many as all the even and odd together. They which in this manner take away eternity from the world, do they not by the same means take away eternity from the creator of the world? . . . And the men that reason thus absurdly are not idiots, but, which makes this absurdity unpardonable, geometri r as, and such as take upon them to be judges, impertinent, but severe judges of other men's demonstrations."

The reference to odd and even numbers doubtless arose from his contact with Galilean thought. While sojourning on the Continent, he had gone to see Galileo, then a prisoner. Hobbes thought he had effected the duplication of the cube and the squaring of the circle. On this matter he became involved in a heated controversy with the algebraist, John Wallis. The aged Hobbes was no match against young Wallis on mathematical questions. When the mathematical works of Wallis were being brought out, Wallis refused to allow his controversial matter against Hobbes to be incorporated in them.[1] Whether the whole is greater than a part was an issue touched upon during this dispute. Hobbes said to Wallis: "All this arguing of infinities is but the ambition of school boys." It cannot be said that Hobbes made any real contribution to a deeper understanding of the "Achilles" or any of Zeno's other arguments on motion. His objection to the dictum, "whatever may be divided into parts infinite in number, the same is infinite," is no new contribution; Aristotle had advanced that far. How Achilles caught the turtle is beyond comprehension through our sensual imagination; Hobbes nowhere explains this inability. However, he does touch upon the concept of a limit in his controversy with Wallis. Hobbes charged that some of the principles of the professors are "void of sense"; one of those principles being, "that a quantity may grow less and less eternally, so as at last to be equal to another quantity; or, which is all one, that there is a last in eternity."[2]

A GENERAL FORMULA FOR THE VALUATION OF SECURITIES.[3]

By JAMES W. GLOVER, University of Michigan.

The object of this paper is to derive a formula for the valuation of a very general type of securities. The security is redeemed in r equal installments at intervals of t years, the first redemption being made after f years. The annual rate of dividend is g payable in m installments, and the security is purchased to realize the investor a nominal rate of interest j with frequency of conversion m.

[1] A full account of the controversy between Hobbes and Wallis is given in Croom Robertson's *Hobbes*, pp. 167–185.

[2] *The English Works of Thomas Hobbes*, Vol. 7, p. 186.

[3] Read before the Chicago Section of the American Mathematical Society, April, 1912. Those unfamiliar with the notation and functions employed in the theory of compound interest may consult *Text-Book of the Institute of Actuaries*, Part I, by Ralph Todhunter; *The Mathematical Theory of Investment*, by Ernest B. Skinner; *Bulletin of the Department of Agriculture*, No. 136, on Highway Bonds, by Laurence I. Hewes and James W. Glover.

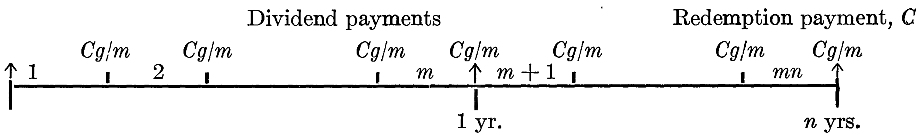
In order to derive this formula we first consider the simple case of a bond redeemed in one payment when the annual rate of dividend is g, payable in m installments, each equal to g/m, and the valuation is made at a nominal rate of interest j with frequency of conversion m.

Let C be the amount to be redeemed after n years.

Let g be the annual rate of dividend, payable in m equal installments g/m, per unit of the redemption fund C.

Let j be the nominal rate of interest, with frequency of conversion m, to be employed in the valuation.

Let A be the value of, or bid upon, the security.



Referring to the figure we see that the value of the security consists of two parts:

1. A series of dividend payments of Cg/m at the end of each mth part of the year for n years, and

2. The sum C to be redeemed at the end of n years.

The first part may be regarded as an immediate annuity-certain with payments of Cg/m per interval and running for mn intervals. Since interest is at the rate j/m per interval, the present value of this annuity is

$$a_{\overline{mn}|} \cdot Cg/m = \frac{1 - v^{mn}}{j/m} \cdot Cg/m = C(1 - v^{mn})g/j, \quad \text{where} \quad v = 1/(1 + j/m).$$

The present value of C, due in n years, or at the end of mn intervals, is Cv^{mn}, hence the present value of the security is

$$A = Cv^{mn} + (g/j)(C - Cv^{mn}), \tag{1}$$

where the function v^{mn} is to be taken at the rate j/m.

We are now prepared to consider the more general problem of valuing a security of the following nature:

1. The security is redeemed in r equal installments.

2. The first redemption payment is made at the end of f years.

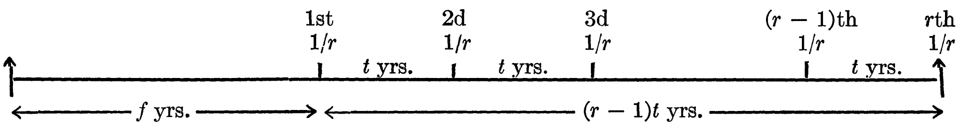
3. The remaining $(r - 1)$ redemption payments are made at intervals of t years.

4. The annual rate of dividend is g and dividends are paid in equal installments at the ends of the m equal intervals into which the year is divided.

5. The security is valued at the nominal rate $j_{(m)}$.

We proceed to find the present value A of a security of the above type whose total redemption fund is unity. Since the unit fund is redeemed in r equal installments each one will be $1/r$; the following figure illustrates the nature of the security.

Redemption payments.



It is evident that the total unit loan may be regarded as made up of r separate loans, each of $1/r$, and to determine its present value we need only to find the values of these several partial loans and add them together. For this purpose we set forth the following scheme:

No. of Loan.	Present Value.	Due or Matures in Years.	Amount Maturing per Unit of Total Sum to be Redeemed.
1	A_1	$n_1 = f$	$C_1 = 1/r$
2	A_2	$n_2 = f + t$	$C_2 = 1/r$
3	A_3	$n_3 = f + 2t$	$C_3 = 1/r$
.	.	. .	.
.	.	. .	.
.	.	. .	.
r	A_r	$n_r = f + (r - 1)t$	$C_r = 1/r$

Employing formula (1) for each one of these loans we have the present values as follows:

$$A_1 = \frac{1}{r}\, v^{mf} + (g/j)\left[\frac{1}{r} - \frac{1}{r}\, v^{mf}\right],$$

$$A_2 = \frac{1}{r}\, v^{m(f+t)} + (g/j)\left[\frac{1}{r} - \frac{1}{r}\, v^{m(f+t)}\right],$$

$$A_3 = \frac{1}{r}\, v^{m(f+2t)} + (g/j)\left[\frac{1}{r} - \frac{1}{r}\, v^{m(f+2t)}\right],$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$A_r = \frac{1}{r}\, v^{m(f+\overline{r-1}t)} + (g/j)\left[\frac{1}{r} - \frac{1}{r}\, v^{m(f+\overline{r-1}t)}\right],$$

all being at rate j/m.

Adding the left-hand column we have A, the present value per unit of the total sum to be redeemed. The first column on the right is a geometric series whose sum is $v^{mf}(1 - v^{mtr})/(1 - v^{mt})$, apart from the factor $1/r$, hence

$$A = \frac{1}{r} \cdot v^{mf} \cdot \frac{1 - v^{mtr}}{1 - v^{mt}} + (g/j)\left[1 - \frac{1}{r} \cdot v^{mf} \cdot \frac{1 - v^{mtr}}{1 - v^{mt}}\right].$$

For convenience, letting

$$z = \frac{1}{r} \cdot v^{mf} \cdot \frac{1 - v^{mtr}}{1 - v^{mt}},$$

the preceding formula may be written

$$A = z + (g/j)(1 - z),$$

whence the *premium* $k = A - 1$ takes the form

$$k = (1 - z)(g - j)/j. \tag{2}$$

Since z is the present value of a number of quantities whose sum is unity, $1-z$ must be positive, and formula (2) shows that the premium k is positive or negative according as the rate of dividend g is greater or less than the rate of interest j desired to be realized by the investor. When k is positive the security is said to be bought at a *premium*, when negative, at a *discount*.

The formula for the premium k on the unit loan expressed in terms of the present value or v-function is:

$$k = \left[1 - \frac{v^{mf}}{r} \cdot \frac{1 - v^{mtr}}{1 - v^{mt}}\right](g - j)/j, \qquad \text{at rate } j/m. \tag{3}$$

The most frequent case in practice is when $m = 2$. Formula (3) then becomes

$$k = \left[1 - \frac{v^{2f}}{r} \cdot \frac{1 - v^{2tr}}{1 - v^{2t}}\right](g - j)/j, \qquad \text{at rate } j/2. \tag{4}$$

These formulas may be modified somewhat since

$$z = \frac{v^{mf}}{r} \cdot \frac{1 - v^{mtr}}{1 - v^{mt}} = \frac{v^{mf}}{r} \cdot \frac{\dfrac{1 - v^{mtr}}{j/m}}{\dfrac{1 - v^{mt}}{j/m}} = \frac{v^{mf}}{r} \cdot \frac{a_{\overline{mtr}|}}{a_{\overline{mt}|}},$$

where $a_{\overline{n}|}$ is the present value of an immediate annuity-certain and one of the usually tabulated interest functions. We have then

$$k = [1 - v^{mf}a_{\overline{mtr}|}/ra_{\overline{mt}|}](g - j)/j, \qquad \text{at rate } j/m, \tag{5}$$

and in the special case when $m = 2$,

$$k = [1 - v^{2f}a_{\overline{2tr}|}/ra_{\overline{mt}|}](g - j)/j, \qquad \text{at rate } j/2. \tag{6}$$

It may be found desirable to express the value of the premium k in terms of the function $a_{\overline{n}|}$ and this can be accomplished as follows:

$$z = \frac{v^{mf}}{r} \cdot \frac{1 - v^{mtr}}{1 - v^{mt}} = \frac{1}{r} \cdot \frac{v^{mf} - v^{m(f+tr)}}{1 - v^{mt}}$$

$$= \frac{1}{r} \cdot \frac{\dfrac{1 - v^{m(f+tr)}}{j/m} - \dfrac{1 - v^{mf}}{j/m}}{\dfrac{1 - v^{mt}}{j/m}} = \frac{1}{r} \cdot \frac{a_{\overline{m(f+tr)}|} - a_{\overline{mf}|}}{a_{\overline{mt}|}},$$

where all the annuities are to be taken at rate j/m.

This leads to the formula

$$(7) \qquad k = \left[1 - \frac{a_{\overline{m(f+tr)}|} - a_{\overline{mf}|}}{r a_{\overline{mt}|}}\right](g - j)/j, \qquad \text{at rate } j/m,$$

and for the important practical case $m = 2$,

$$(8) \qquad k = \left[1 - \frac{a_{\overline{2(f+tr)}|} - a_{\overline{2f}|}}{r a_{\overline{2t}|}}\right](g - j)/j, \qquad \text{at rate } j/2.$$

Believing that the readers of the MONTHLY may be interested in the operation of a bond loan of this character we give the following example.

What is the premium on \$100,000 highway bonds, interest 5% payable semiannually, dated January 1, 1914, maturing \$50,000 January 1, 1917, and \$50,000 January 1, 1919, to net the purchaser 4 per cent. compounded semiannually?

Here $f = 3$, $r = 2$, $t = 2$, $m = 2$, $g = .05$, $j = .04$, hence $m(f + tr) = 14$, $mf = 6$, $mt = 4$. Consulting a table[1] of annuities, $a_{\overline{n}|}$, with 2 per cent. as the rate of interest, and employing formula (8), the numerical work may be outlined as follows:

$$a_{\overline{14}|} = 12.10624877 \qquad (1)$$
$$a_{\overline{6}|} = 5.60143089 \qquad (2)$$
$$a_{\overline{14}|} - a_{\overline{6}|} = 6.50481788 \qquad (3)$$
$$(3) \div 2 = 3.25240894 \qquad (4)$$
$$a_{\overline{4}|} = 3.80772870 \qquad (5)$$
$$(4) \div (5) = .85415984 \qquad (6)$$
$$\text{Complement of } (6) = 1 - (6) = .14584016 \qquad (7) = \text{first factor}$$
$$(.05 - .04)/.04 = .25 \qquad (8) = \text{second factor}$$
$$k = (7) \times (8) = .03646004$$

The bid on one dollar is 1.03646004, hence the bid on the entire issue is \$103,646.004. The progress of the loan is indicated in the following schedule.

(1) Interval.	(2) Year.	(3) Book Value or Principal at Beginning of Interval.	(4) Semiannual Interest of 2%.	(5) Semiannual Dividend of $2\frac{1}{2}$% on Bonds.	(6) Amortization of Premium at End of Interval.	(7) Redemption Payment at End of Interval.
1	$\frac{1}{2}$	\$103,646.00	\$2,072.92	\$2,500.00	\$427.08	0.00
2	1	103,218.92	2,064.38	2,500.00	435.62	0.00
3	$1\frac{1}{2}$	102,783.30	2,055.67	2,500.00	444.33	0.00
4	2	102,338.97	2,046.78	2,500.00	453.22	0.00
5	$2\frac{1}{2}$	101,885.75	2,037.72	2,500.00	462.28	0.00
6	3	101,423.47	2,028.47	2,500.00	471.53	\$50,000.00
7	$3\frac{1}{2}$	50,951.94	1,019.04	1,250.00	230.96	0.00
8	4	50,720.98	1,014.42	1,250.00	235.58	0.00
9	$4\frac{1}{2}$	50,485.40	1,009.71	1,250.00	240.29	0.00
10	5	50,245.11	1,004.89	1,250.00	245.11	50,000.00
Totals		817,699.84	16,354.00	20,000.00	3,646.00	100,000.00

[1] Compound Interest and Annuity Calculations with Tables, W. M. J. Werker.

By adding the several columns in this schedule various checks are obtained which are too evident to need comment. Attention is called to the fact, however, that it is not necessary to construct the whole schedule in order to determine an item in a given row and column. For example the fifth item in column three, \$101,885.75, representing the present value of the outstanding security at the beginning of the fifth interval, can be calculated directly by making the proper substitutions in formula (8). When this is known the other items in the same row can be determined at once.

There are several special cases of formula (7) which deserve mention. The most common type of serial bond bears semiannual dividends and is redeemed in n equal *annual* installments, the first of which is paid at the end of the first year. In this case $f = t = 1$, $r = n$, $m = 2$, and

$$k = \left[1 - \frac{a_{\overline{2n+2}|} - a_{\overline{2}|}}{n a_{\overline{2}|}}\right](g - j)/j, \qquad \text{at rate } j/2. \tag{9}$$

Since

$$a_{\overline{2n+2}|} = a_{\overline{2}|} + v^2 a_{\overline{2n}|} \quad \text{and} \quad v^2/a_{\overline{2}|} = 1/s_{\overline{2}|},$$

formula (9) may be written

$$k = \left[1 - \frac{a_{\overline{2n}|}}{n s_{\overline{2}|}}\right](g - j)/j, \qquad \text{at rate } j/2. \tag{10}$$

When the serial bond is like the preceding except that dividends are payable and interest is convertible annually we have $f = t = 1$, $r = n$, $m = 1$, and

$$k = \left[1 - \frac{a_{\overline{n+1}|} - a_{\overline{1}|}}{n a_{\overline{1}|}}\right](g - j)/j, \qquad \text{at rate } j/1.$$

In this case the nominal rate $j_{(1)}$ equals the effective rate i and, since $a_{\overline{n+1}|} = a_{\overline{1}|}\,(1 + a_{\overline{n}|})$, the formula reduces to

$$k = \left[1 - \frac{a_{\overline{n}|}}{n}\right](g - i)/i. \tag{11}$$

When the bond is redeemed in a single installment at the end of its term, say n years, we have, $f = n$, $r = 1$, and formula (7) reduces to

$$k = \left[1 - \frac{a_{\overline{m(n+t)}|} - a_{\overline{mn}|}}{a_{\overline{mt}|}}\right](g - j)/j,$$

but since

$$a_{\overline{mn+mt}|} = a_{\overline{mn}|} + v^{mn} \cdot a_{\overline{mt}|},$$

the formula for the premium may be written

$$k = \left[\frac{1 - v^{mn}}{j/m}\right](g - j)/m, \qquad \text{at rate } j/m$$

or finally,

$$k = a_{\overline{mn}|}\,(g - j)/m, \qquad \text{at rate } j/m. \tag{12}$$

When, as is usually the case, dividends on the bond are paid semiannually and it is valued to net the purchaser a nominal rate j convertible twice a year, formula (12) becomes

$$k = a_{\overline{2n}|}\,(g - j)/2, \qquad \text{at rate } j/2. \tag{13}$$

In the simplest of all cases, dividends payable annually at rate g, interest compounded annually at the effective rate i, and the bond maturing in one sum at the end of n years, (12) reduces to the well-known form

$$k = a_{\overline{n}|}\,(g - i). \tag{14}$$

This formula admits of a simple interpretation because it states that the premium per unit of the sum to be redeemed is equal to the present value of an annuity whose annual rent is equal to the excess of the rate of dividend over the rate of interest desired to be realized by the purchaser. I may add that practically all the formulas in this paper admit of a direct interpretation. The interpretation of the final formula usually suggests a simple method of deriving it by general considerations and always throws a great deal of light upon the nature of the problem. It is not my purpose, however, to enter at this time upon the subject of interpretation of interest formulas.

QUESTIONS AND DISCUSSIONS.

EDITED BY U. G. MITCHELL.

NEW QUESTIONS.

24. The following facts are significant:

(1) The New England Association of Mathematics Teachers has appointed a committee "to investigate the current criticisms of high school mathematics."

(2) A committee of the Council of the American Mathematical Society has under consideration the question "whether any action is desirable on the part of the Society in the matter of the movement against mathematics in the schools."

(3) At the recent meeting in Cincinnati of the National Education Association an iconoclastic discussion on the topic: "Can algebra and geometry be reorganized so as to justify their retention for high school pupils not likely to enter technical schools?" aroused approbation and applause. An outline of the remarks by one of the speakers will be printed in this column next month.

In view of these facts what should be done by those who believe in the value of mathematics as a general high school study?

REPLIES.

9. What is the present state of experience with coördinated courses in high school mathematics? What contribution does this promise to the development of mathematics teaching in high schools? What about the corresponding matters in college mathematics? (*Note.*—An individual correspondent need not answer all the questions in number 9; it is sufficient if he answers only one.)

REPLY BY ROY CUMINS, Columbia University, N. Y.

At present there exists in the United States a decided movement toward breaking down the barriers that have hitherto kept separate the various branches

of secondary mathematics. These branches are, let us say, algebra, geometry, trigonometry, and an extension of arithmetic; but this discussion will deal primarily with algebra and geometry.

At least three words are popularly employed to designate this movement. They are fusion, unification, and correlation. Much of the difference of opinion that has been expressed with regard to the movement grows out of the fact that these words are not defined in their relation to the teaching and the subject-matter of secondary mathematics. Moreover, a person who might favor correlation opposes, say, unification, without mentioning correlation, and thus is inferentially antagonistic to all phases of the movement.

Let us inquire, then, concerning the meaning of the three words as applied to secondary mathematics. In the first place they concern not so much the content of the subject-matter of algebra and geometry as they do the arrangement and treatment of this subject-matter. This statement cannot be pushed too far, as the texts bearing the names of Myers, Cobb, Long and Brenke, and Short and Elson touch more intimately the actual out-of-school life of the pupil and find the source of more of their applications in such fields as physics, mechanics, biology, engineering, design, architecture, and the trades and industries than did the texts commonly used a decade ago. But the basis for defining our terms must chiefly be sought elsewhere.

Of the three words, "fusion" is certainly the strongest. It may mean that the subject-matter of algebra and that of geometry are to be "fused" in the way in which metals are fused to form an alloy or amalgam. Or, if we do not go quite so far as that, it means fusion to the extent to which the theorems of plane geometry are fused in the ordinary high-school text in geometry. Finally, the word may signify only the fusion which is exhibited in the ordinary high-school text in algebra. Certainly it should mean at least this much.

At the other end of the list, "correlation" may mean almost anything. No doubt we have always been correlating the subject-matter of algebra and geometry to some extent. It would be very difficult and unusual, although not impossible, to teach algebra for a year or a year and a half and then teach geometry for a year without employing in some measure the symbolism, facts, and tools of algebra.

Of course, what the advocates of the present movement wish is close and frequent correlation between the sciences of function and form. They wish coöperation and the penetration of the water-tight (may we say thought-tight?) barrier that we must admit has too effectually kept apart algebra and geometry.

Between the two terms spoken of comes "unity." To me it implies less than "fusion" and more than "correlation." As used in theme writing in my school days, it meant that everything in the subject-matter should bear on the central thought, the purpose, the organizing principle. Nothing irrelevant is to be admitted. To determine what should be admitted into a unified course in secondary mathematics, it would therefore be necessary to decide first upon an organizing principle for secondary mathematics. That task remains to be done.

Many advocates of the plan to relate more closely algebra and geometry continually cite what other countries, especially Germany and France, have done in this regard. For what is said here concerning the teaching of mathematics in these countries I am indebted to the writings of Professors Farrington, Myers, and J. W. A. Young, and to instruction by Professor C. B. Upton.

In France, says Professor Farrington, "Throughout the mathematics courses one is impressed with the intimate relations existing among the various subjects. Arithmetic is not carried to a certain point, there to give way to algebra, in its turn, perhaps, to be supplemented by geometry, but from the fifth form in one division and from the fourth form in another, at least two subjects are run conjointly." So what exists in France is not fusion or unification but parallelism of treatment and correlation. From the time algebra, geometry, and science are begun they are carried along side by side to the end of the course.

Professor Young summarizes the work in Prussia as follows: "In the German schools the subject of study is *mathematics*, and its various branches are studied side by side. At no time is only one subject being studied with the exception of the two lowest years, in which only arithmetic is taken up."

Not long ago, Professor Klein, in a conversation with Professor Upton, expressed his amusement at a seemingly prevalent idea in America that the Germans no longer teach algebra and geometry as such but fuse the two into a course in mathematics. What he says they do is to teach each of the subjects as such and let them help each other whenever they can. Clearly here again is parallelism of treatment with considerable correlation.

Professor Myers says, "The dominant phase of European secondary mathematics is of the mixed, or parallel, or quick alternation type." This does not sound like fusion or unification, as I have defined them.

What do American texts do in "fusing" algebra and geometry? For the most part they do not fuse the two at all, not even the newest texts. Their common plan is to give a chapter or two on algebra, then devote the next pages to geometry, with little or no reference to the algebra that preceded; and then follow with another chapter on algebra that is almost entirely unrelated to the geometry that preceded. (I have not considered the Myers texts in this regard, as they are now being revised, nor the Cobb text, as it is not to my knowledge extensively used in high schools.)

Why should we give parallel courses in geometry and algebra instead of the traditional tandem courses? As it is, one class contends that algebra should precede, while another asks that geometry be taught first. To a high school freshman the beginnings of geometry are certainly more simple, concrete, and familiar than those of algebra. They relate to things that the pupil actually sees out in the world or can construct for himself. On the other hand, the easier parts of algebra are far simpler than the more difficult parts of geometry. If the two were taught simultaneously, each could be of great assistance to the other and correlation could be much more easily effected than is now done. In the third place, the pupil would not rush at high pressure over each subject as he

now does. A boy in a German school, for example, studies algebra for six years and geometry for eight years. Time for the maturing of thought and for the effects of retroactive inhibition are thus afforded. Pedagogically, then, we should have parallelism of treatment.

Furthermore, as Professor Young has pointed out, we are not compelled to conduct an experiment to demonstrate the practicability of our theory. France and Germany have done that for us and they tell us with assurance that a great gain has been accomplished. What I favor, therefore, is that both algebra and geometry be begun early, with geometry first, that the study of each be extended over a far greater time than is done at present, and that we do our best in bringing the subject-matter, symbolism, and methods of each subject to assist the other. This should be called parallelism with all possible correlation.

19. How many known proofs are there of the proposition that the square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides? Where are the proofs to be found?

Reply by C. E. Horne, Westminster College, Colorado.

As a possible reply to the above question I should like to submit the following:

Jury Wipper, *Sechsundvierzig Beweise des Pythagoräischen Lehrsatzes.* Aus dem Russischen von F. Graap, Leipzig, 1880.

If there is anything more complete than this work, I shall be glad to learn of it.

(*Note.*—It may be worth while to mention also, in this connection, the earlier and less complete work of Ignaz Hoffman, *Der Pythagoräischen Lehrsatz mit 32 Beweisen,* Mainz, 1819.—Editor.)

Reply by H. E. Slaught, The University of Chicago.

It may be of interest that the first class in geometry which I taught, in 1883, became "Pythagorean crazy," and that we elaborated over forty proofs, each being carefully copied on stiff cardboard and all bound together and deposited in the archives of the school, Peddie Institute, Heightstown, N. J. I hope they are still preserved.

22. What can the colleges do toward improving the teaching of mathematics in secondary schools?

Reply by Charles N. Moore, University of Cincinnati.

It is evidently desirable that colleges and universities should do everything they reasonably can to improve the teaching of preparatory subjects in the secondary and primary schools in their localities. Not only is this their duty from the broad viewpoint of the relation of such institutions to society; but, from the narrower viewpoint of benefits to the institutions themselves, any results achieved in this direction will react in a stimulating way upon the college or university. Consequently it seems worth while to give here a brief account of what the department of mathematics at the University of Cincinnati is and has been doing in that direction for the past fifteen years.

The University of Cincinnati, being a municipal institution, has always made a special effort to coöperate as much as possible with all municipal activities,

and particularly with all activities in other subdivisions of the public educational system of the city. In line with this general policy, courses especially designed for those who are actively engaged in teaching have been given by all the departments in the late afternoon hours and on Saturdays. These courses have been highly appreciated and well attended by teachers in the city and in the neighboring towns.

In the department of mathematics at least one such course, and sometimes three or more, have been given each year during the past fifteen years. These courses have ranged from freshman college courses to advanced graduate courses, in order to meet the needs and interests of as many teachers in the vicinity as possible. Very few of them were definitely method courses, for it is a conviction of the department, based on its own experiences, that most teachers of primary or secondary subjects who have had some experience in actual teaching will profit more from additional information in their subject than they will from specific training in methods. Of course incidental discussion of pedagogical questions is valuable, and we have always kept that in mind. But our principal effort has been to give the teachers in the preparatory schools of our vicinity a wider mathematical horizon.

As a result of our efforts in this direction almost all the younger teachers of mathematics in the Cincinnati high schools have taken one or more courses in mathematics at the University, and quite a number of them have taken the master's degree in mathematics. Some of these latter did all their mathematical study from freshman college work through to the master's degree in connection with these courses given especially for teachers. Also quite a number of teachers in the primary schools, including several who are now principals of schools in the city, and a number of teachers in preparatory schools of neighboring towns, have taken one or more courses in mathematics.

Another feature of the work done at the University of Cincinnati to bring about closer coöperation with the teachers in the preparatory schools is an annual gathering at the University of teachers in the accredited secondary schools. Those in attendance are grouped into sections according to the subjects in which their chief interest lies, and pedagogical questions relating to secondary schools are discussed. The members of the different departments of the university make it a point to attend the sectional meetings in which their subjects are discussed, and not only participate in the discussion but frequently give one of the papers of the day. The papers presented in the mathematical section and the ensuing discussions have been found to be extremely interesting and helpful to all those attending, and there is no doubt that these annual meetings have done much to increase the feeling of a community of interest between teachers in the university and those in the secondary field.

The chief apparent effect, as noticed by the department of mathematics, of this steady effort to do something toward the further training of teachers in the preparatory schools has been to unify the teaching of mathematics in this vicinity and to bring about a closer coöperation among all those engaged in such teaching,

whether the subject taught be arithmetic or the theory of functions. We are thoroughly convinced that the value of this unification and coöperation makes the effort put forth to obtain it seem very slight in comparison. We are also convinced that there are other beneficial effects of this effort which are less apparent but not less important. And, finally, we believe that any college or university that has done little or no work of this sort will find it highly worth while to increase its activity in this direction.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY, University of Minnesota.

Analytic Geometry. By L. WAYLAND DOWLING and F. E. TURNEAURE. Henry Holt and Co., New York, 1914. xi + 266 pages.

A chief feature of several recent texts on analytics is the emphasis of the general idea of a function and its graph rather than of the theory of conics. This text retains that emphasis to a large extent. After 25 pages devoted to chapters on "Systems of Coördinates" (I) and "Directed Segments and Areas of Plane Figures" (II), we find 65 pages of discussion of "Functions and their Graphic Representation" (III), "Loci and their Equations" (IV), and "Equations and their Loci" (V), the last chapter including "Transformation of Coördinates." In Chapter III first methods of graphing functions are given with illustrations from algebraic and transcendental functions in rectangular and polar coördinates. The equation of a locus is defined in Chapter IV and the standard equations of straight lines, the conics and Cassinian ovals are derived for both coördinate systems. Chapter V gives methods of discussing an equation with numerous examples. It is not until Chapter VI, "Loci of First Order," beginning on page 98, that we find a systematic treatment of the straight line. After what has preceded, 10 pages suffices. Then in Chapter VII, "Loci of Second Order, Equations in Standard Form," we find, compassed in 34 pages, a fairly complete elementary treatment of the conics, including "Poles and Polars" and "Systems of Conics." The next chapter treats the general equation of second degree.

In accord with another modern development, we find (included in Chapter IX, "Loci of Higher Order and Other Loci") a twelve page discussion of "Empirical Equations and their Loci." This subject, on account of its importance in applications of mathematics to the sciences, seems destined to become an essential part of a good course in analytics. The presentation of the authors, which includes the use of logarithmic coördinate paper, is excellent.

Following these chapters on plane analytics is a brief treatment (about 50 pages) of solid analytics. There is little of novelty in this part of the book.

The book as a whole impresses one very favorably. The general order of presentation is excellently adapted to give the student a real appreciation of the power and beauty of analytic geometry, and also the ability to use it. The tendency to lay a little more stress than usual on the geometrical aspect of the

subject, especially in the chapter on "Loci and their Equations," by numerous figures and by various methods of constructing some of the loci, will be welcomed by many teachers.

There are several matters of detail, however, which one may criticise. Exceptional cases of various sorts are, for example, quite generally ignored. Thus in discussing the two-point form of the equation of a straight line, the exceptional cases of lines parallel to either coördinate axis receive no mention. In giving the slope forms, no mention is made of lines parallel to the y-axis. And all lines through the origin are ignored in connection with the intercept and normal forms. The determinant form of the equation of a plane through three points is said to be linear without mentioning the exceptional case arising when the points are collinear. Likewise no account is taken of exceptional cases in the discussion of pencils of lines or of conics.

While the general idea of Chapter IV, "Loci and their Equations," appeals to me, the logic seems unsatisfactory in two respects. By definition, "The equation of the locus of a point is an equation in the variables x and y which is satisfied by the coördinates of every point on the locus; and conversely. . . ." The "*and conversely*" is subsequently neglected without comment in deriving equations except in the case of the circle. This, of course, invalidates the proof, for instance, that the locus of every equation of first degree is a straight line. The proof of the "*and conversely*" for a straight line is as difficult as the direct, and the omission seems hardly excusable. The second criticism is on the expression, "*the* equation of the locus . . ." without comment on the first "*the*." There is an infinite number of equations satisfying the prescribed conditions in general, and some remark is apparently necessary before one of them may be designated as "*the* equation." For example, the following are all equations of the same real locus:

$$x^2 + y^2 = 4; \quad x^4 + 2x^2y^2 + y^4 = 16;$$

$$9x^2 + 9y^2 = 36; \quad x^4 + x^2y^2 + 4y^4 = 16.$$

At least a footnote of explanation seems desirable.

It is further noted that the proof of the formula for the distance from a line to a point (p. 104) is invalid for some cases (e. g., for Fig. 66, p. 105).

As a minor point for criticism, one is surprised to find that the formula for the distance between two points is not given for rectangular cartesian coördinates as distinct from oblique coördinates. Also no reason for deriving formulas for the mid-point of a line segment by a method different from that required to find the point of division in a given ratio, r, is apparent since the former is a simple corollary of the latter. And although the latter is given in the usual manner, it is open to a certain criticism. A point P is found on P_1P_2 such that $P_1P/PP_2 = r$; its coördinates are

$$x = \frac{x_1 + rx_2}{1 + r}; \quad y = \frac{y_1 + ry_2}{1 + r}.$$

There is such a point for all values of r *except* $r = -1$; and by taking all values of r we get all points on the line P_1P_2 *except* P_2. These exceptional cases are avoided and we get simpler formulas, which are more easily derived, if we determine P such that $P_1P/P_1P_2 = r$ as follows; namely[1]

$$x = x_1 + r\cdot\Delta x = x_1 + r(x_2 - x_1), \qquad y = y_1 + r\cdot\Delta y = y_1 + r(y_2 - y_1).$$

To sum up, although marred by several inaccuracies in detail, the book as a whole is very good, thoroughly modern, and includes much in a small compass.

E. J. MOULTON.

NORTHWESTERN UNIVERSITY.

The Algebra of Logic. By LOUIS COUTURAT. Authorized translation by LYDIA G. ROBINSON, with a preface by PHILIP E. B. JOURDAIN. The Open Court Publishing Co., Chicago, 1914. xiv + 98 pages. $1.50.

This volume is a translation of volume number 24 in the Gauthier-Villars collection under the general title Scientia. These very useful manuals run about 100 pages each, in two series, the present number belonging to the physico-mathematical series. The translation has been well done although it is not exactly a literal rendering of the French style of sentence. A bibliography has been inserted and the preface presents, in brief form, a good history of the development of symbolic logic. These additions are distinct improvements and will make the book more useful.

The algebra of logic, as its name indicates, is a mathematical treatment of logic. It is not a new species of numerical calculation by symbols, but is a treatment of the field of logical notions by introducing symbols which have to be combined according to definite laws. It is an application of the methods of mathematics to logic, and is just as much applied mathematics as is mechanics. The only peculiarity in the situation is that in the reasoning processes employed, the symbols and laws relate to the reasoning process itself. The field of study in logic consists of classes of objects of the mind as marked by some distinguishing quality and the relations between such classes; assertions regarding mental objects and their relations; and relations themselves as objects with relations to each other. For example we may study the class of triangles that are distinguished as isosceles, and the class that are distinguished as having two equal angles. The identification of the two classes is done by logical processes. If we consider the problem of identification of classes of objects in general we are studying logic. Just as no one can get along in the world without using arithmetic consciously or unconsciously, so no one can get along without reasoning. If he becomes interested in the processes of arithmetic and the study of identities between numerical expressions in general, then he begins to study algebra. So too if he becomes interested in the process of reasoning he begins to study logic, and the most efficient way to do this is to apply the mathematical method; that is to say, to devise a system of symbols that will take the place of the mental objects under study, to parallel the mental combinations of these objects with

[1] These formulas are given, for example, by Ziwet and Hopkins, *Analytic Geometry*, p. 8.

combinations of the symbols, and to state as laws of operation the results of the mental transition from object to object.

Consider, for example, the process of reasoning called the syllogism. In its most simple form this consists in the succession of statements:

If objects A defined in any way have the quality B, and if objects that have the quality B must have the quality C, then the objects called A must have the quality C. We state this symbolically

$$(A < B)(B < C) < (A < C).$$

We are in a much better position to discuss the syllogism from this visualization of it in symbols than if we try to discuss it in words and in sentences. We can indeed reduce the manipulation of our logical formulæ to a few very simple rules which not only enable us to produce the usual results of logic, but we can solve very much more difficult questions. For examples see Jevons' *Principles of Science*, or his *Studies in Deductive Logic*.

We can do more however, for if A, B, C above are not simple classes but are propositions, we have a formula relating them in precisely the same manner as the syllogism above, and all our formulæ become susceptible of another interpretation. We can thus reduce our two-fold logic in the main to a single treatment, which is a distinct gain. The author does this throughout the book. The formulæ developed are interesting in showing what can be done in the discussion of general questions. However it should be remembered that for any concrete case, the fundamental rules of the symbolism are all one needs to solve the problem.

A word is in place perhaps as to why one should study symbolic logic at all. The best answer is that it is desirable for precisely the reasons that one should study algebra, or theory of groups. While everyone possibly is gifted with correct reasoning power, at least so long as the problem is simple, it is often not possible to carry an intricate set of premises in the mind and deduce the correct conclusion. Nor is it possible easily to solve the reverse cases and say, for example, what would be the simplest set of hypotheses that would make certain propositions follow as consequences. With the machinery of symbolic logic this becomes quite manageable.

As an example we mention the famous problem of Venn stated thus:

The members of a board are either bondholders or shareholders, but not both; all the bondholders are on the board. What conclusions can be drawn?

As an example of inverse logical problems we may quote the following: What are the premises that enable us to conclude that objects with the quality C do not have the quality E, and that objects without the quality C must have the quality E, do not have the quality A, and also either do not have the quality B or else have the quality B but do not have the quality D?

These simple problems illustrate the character of work that can be handled easily by symbolic logic but not easily in the ordinary way. They exemplify in their solution also the fact that the syllogism is only one of many ways of

reasoning. This however we have not space to enter into, and it should be followed up after reading the book before us, in some of the more extended treatises mentioned in the bibliography in this book. Every student of mathematics ought to read through some book on logic, and the present one is an admirable introduction to the subject.

JAMES BYRNIE SHAW.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

PROBLEMS FOR SOLUTION.

Note.—An additional supply of good live problems is desired, especially in algebra, calculus, and mechanics.—EDITORS.

ALGEBRA.

When this issue was made up, solutions had been received for numbers 424–427

428. Proposed by FRANK IRWIN, University of California.

If the roots of the equation

$$x^n - na_1x^{n-1} + \binom{n}{2} a_2x^{n-2} + \cdots = 0$$

are all real, the condition that they should all be equal is $a_1^2 = a_2$. A proof of the sufficiency of the condition is readily obtained from a consideration of derivatives. A proof is desired not based on such considerations.

429. Proposed by C. N. SCHMALL, New York City.

It is given that d_1, d_2, d_3, are the greatest common divisors of y and z, z and x, x and y, respectively; also that m_1, m_2, m_3, are the least common multiples of the same pairs of numbers. If d and m are the greatest common divisor and least common multiple, respectively, of x, y, and z, show that

$$\frac{m}{d} = \left(\frac{m_1m_2m_3}{d_1d_2d_3}\right)^{1/2}.$$

430. Proposed by V. M. SPUNAR, Chicago, Illinois.

Solve the equations algebraically and also graphically: $x^y + y^x = xy, \qquad x^x + y^y = x + y.$

GEOMETRY.

When this issue was made up, solutions had been received for numbers 452–454

457. Proposed by NATHAN ALTSHILLER, University of Washington.

AB and AC are respectively a diameter and a chord of a circle whose center is O. The lines joining B to the extremities of the diameter perpendicular to AC, meet AC in the points M, N. Express the angle MON in terms of the angle CAB.

458. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Given edges l, m, and n of a parallelopiped and the angles a, b, and c which the edges make with one another. Show that, if $s = (a + b + c)/2$, the volume equals

$$2lmn\sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}.$$

459. Proposed by C. N. SCHMALL, New York City.

In a right triangle ABC, right-angled at C, a point F is taken in the side CB and perpendiculars CD and FE are dropped on the hypothenuse AB. Prove $AD \cdot AE + CD \cdot EF = \overline{AC}^2$.

CALCULUS.

When this issue was made up, solutions had been received for numbers 366–377

378. Proposed by ELBERT H. CLARKE, Purdue University.

The area of the curved surface generated by the revolution about OX of the portion of the curve $y = x^n$ which extends from the origin to the point (1, 1) is given by the formula

$$A = 2\pi \int_0^1 x^n \sqrt{1 + n^2x^{2n-2}}dx.$$

Our geometric intuition would tell us that the limit of this area as n becomes infinite is π. Give a strict analytic proof that

$$\lim_{n \doteq \infty} \int_0^1 x^n \sqrt{1 + n^2x^{2n-2}}dx = \tfrac{1}{2}.$$

379. Proposed by C. N. SCHMALL, New York City.

Express the equation of the folium, $x^3 + y^3 = 3axy$, in parametric form and find the area of the loop.

(From E. B. Wilson's *Advanced Calculus*, p. 296, ex. 5.)

MECHANICS.

When this issue was made up, solutions had been received for numbers 297, 301, and 302

303. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

A pile-driver weighing 500 pounds falls through 10 ft. and drives a pile weighing 400 pounds 3 inches into the ground. Show that the average force of the blow is $11{,}111\frac{1}{9}$ pounds.

NUMBER THEORY.

When this issue was made up, solutions had been received for numbers 224, 225, 226, and 229

228. Proposed by HERMON C. KATANIK, Indianapolis, Ind.

Deduce a formula for the difference between any two squares, and thus show that (1) The difference between any two consecutive squares is of the form $2p + 1$; (2) The difference between any two squares is even or odd according to whether they are separated by an odd or even number of squares; (3) The differences of the squares of the consecutive terms of any arithmetic progression form another arithmetic progression.

229. Proposed by WALTER C. EELLS, U. S. Naval Academy.

If p and q are integers and p is prime and positive, find the condition on q that the equation $p^x = qx$ shall have integral solutions, solve for x, and show that for a special value of p it has two solutions for a given q, otherwise only one.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

418. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Form the algebraic equation whose roots are

$$a_1 = 2\cos\left(\frac{2\pi}{15}\right), \quad a_2 = 2\cos\left(\frac{4\pi}{15}\right), \quad a_3 = 2\cos\left(\frac{8\pi}{15}\right), \quad a_4 = 2\cos\left(\frac{14\pi}{15}\right).$$

SOLUTION BY ELIJAH SWIFT, University of Vermont.

The equation, $x^{15} - 1 = 0$, has for its roots

$$r_k = \cos\left(\frac{2k\pi}{15}\right) + i\sin\left(\frac{2k\pi}{15}\right),$$

where $k = 1, 2, 3, \cdots, 15$; $r_k + \frac{1}{r_k} = 2\cos\left(\frac{2\pi k}{15}\right)$. If, then, we divide out all the roots that are not primitive, leaving only $r_1, r_2, r_4, r_7, r_8, r_{11}, r_{13}, r_{14}$, and, noting that $r_{14} = \frac{1}{r_1}$, $r_{13} = \frac{1}{r_2}$, $r_{11} = \frac{1}{r_4}$, $r_8 = \frac{1}{r_7}$, then reduce this equation by the usual method for a reciprocal equation, we shall have the desired equation. This gives us $x^4 - x^3 - 4x^2 + 4x + 1 = 0$.

In a similar manner we can form the equation whose roots are all distinct values of $2\cos\left(\frac{2\pi k}{n}\right)$, where k takes on all integral values, prime to n between and including 1 and $n - 1$.

Also solved by A. M. HARDING and the PROPOSER.

419. Proposed by GEORGE A. OSBORNE, Massachusetts Institute of Technology.

Show that

$$1^5 + 2^5 + 3^5 + \cdots + n^5 + 1^7 + 2^7 + 3^7 + \cdots + n^7 = 2(1 + 2 + 3 + \cdots + n)^4.$$

SOLUTION BY ELMER SCHUYLER, Brooklyn, N. Y.

By induction we must show that

$$(A)\quad \begin{aligned} &1^5 + 2^5 + 3^5 + \cdots + n^5 + (n+1)^5 + 1^7 + 2^7 + 3^7 + \cdots + n^7 \\ &\qquad + (n+1)^7 = 2(1 + 2 + 3 + \cdots + n + (n+1))^4 \end{aligned}$$

Starting with the right member of (A), we have

$$\begin{aligned} &2(1 + 2 + 3 + \cdots + n + \overline{n+1})^4 \\ &= 2\left[\left\{\frac{n(n+1)}{2}\right\}^4 + 4\left\{\frac{n(n+1)}{2}\right\}^3(n+1) + 6\left\{\frac{n(n+1)}{2}\right\}^2(n+1)^2 \right. \\ &\qquad \left. + 4\left\{\frac{n(n+1)}{2}\right\}(n+1)^3 + (n+1)^4\right] \\ &= 2(1 + 2 + 3 + \cdots + n)^4 + (n+1)^4(n^3 + 3n^2 + 4n + 2) \\ &= 2(1 + 2 + 3 + \cdots + n)^4 + (n+1)^5(n^2 + 2n + 2) \\ &= 2(1 + 2 + 3 + \cdots + n)^4 + (n+1)^5 + (n+1)^7, \end{aligned}$$

which, if the given equation holds, reduces to

$$1^5 + 2^5 + 3^5 + \cdots n^5 + (n+1)^5 + 1^7 + 2^7 + 3^7 + \cdots n^7 + (n+1)^7,$$

and this is the left member of (A). Hence the induction is established.

Also solved by GEORGE Y. SOSNOW, WALTER C. EELLS, J. L. RILEY, CLIFFORD N. MILLS, JAMES A. BULLARD, ELBERT H. CLARKE, A. M. HARDING, R. M. MATHEWS, B. F. YANNEY, ELIJAH SWIFT, and S. A. JOFFE.

GEOMETRY.

433. Proposed by R. P. BAKER, University of Iowa.

A transformation of the plane keeping the radius of curvature of all curves invariant is either (1) a real or imaginary motion or reflexion, or (2) not a point transformation.

Note.—By mistake this problem was originally credited to W. H. BUSSEY, instead of to R. P. BAKER, who is the rightful Proposer.—EDITORS.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

Suppose that this is a point transformation transforming points in plane I to points in plane II. If one configuration is transformed into another, the second is called the *image* of the first. The theorem is not necessarily true if only a finite number of points or even a one-dimensional set be transformed, and I shall assume that the transformation is defined over a two-dimensional region of I possessing at least one interior point and over a similar region of II, and that it is one-to-one in these regions. Now the image of a straight line is a straight line and that of a circle an equal circle. Hence the transformation satisfies the conditions given in an article of mine in the *Bulletin of the American Mathematical Society* (2d series, vol. x, 1903–1904, pp. 247 ff.), and hence must be a collineation. If it is to transform a circle into a circle, it must leave the circular points at infinity invariant (or interchange them) and hence is a finite collineation of the form $x' = a_1x + b_1y + c_1$, $y' = a_2x + b_2y + c_2$. Omitting the c's, which corresponds to making a translation, and writing the condition that $x^2 + y^2 = r^2$ is carried into itself, we obtain the equations $a_1^2 + a_2^2 = b_1^2 + b_2^2 = 1$, $a_1b_1 + a_2b_2 = 0$. If we put $a_1 = \cos\alpha$, $a_2 = \sin\alpha$, then we find that $b_1 = \sin\alpha$, $b_2 = -\cos\alpha$, or $b_1 = -\sin\alpha$, $b_2 = \cos\alpha$. Our collineation is then either of the form

$$\left.\begin{aligned} x' &= \cos\alpha x + \sin\alpha y + c_1 \\ y' &= \sin\alpha x - \cos\alpha y + c_2 \end{aligned}\right\},$$

which corresponds to (1) a rotation through the angle α, (2) a translation through a distance c_1 along the x-axis, and a distance $-c_2$ long the y-axis, and (3) a reflexion in the x-axis; or else of the form

$$\left.\begin{aligned} x' &= \cos\alpha x + \sin\alpha y + c_1 \\ y' &= -\sin\alpha x + \cos\alpha y + c_2 \end{aligned}\right\},$$

which corresponds to a rotation followed by a translation.

447. Proposed by HORACE OLSON, Chicago, Ill.

Given the edge of a regular tetrahedron, find the radius of the circumscribed sphere.

SOLUTION BY MRS. ELIZABETH BROWN DAVIS, U. S. Naval Observatory.

Let A-BCD be a regular tetrahedron, whose edge is a. Let the radius of the circumsphere be R. In the $\triangle\ BCD$, let the medians meet in O. Then $BC = a$; $BB' = \dfrac{a\sqrt{3}}{2}$; $BO = \dfrac{2}{3} \times \dfrac{a\sqrt{3}}{2} = \dfrac{a\sqrt{3}}{3}$. Since $\triangle\ BCD$ is equilateral, O is

equidistant from BC, CD, and BD, and hence equidistant from the planes A-BC, A-CD, and A-BD. Hence O lies on the line of intersection of the three planes bisecting the dihedral angles C-AB-D, B-AD-C, and D-AC-B.

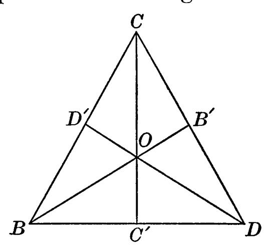


FIG. 1.

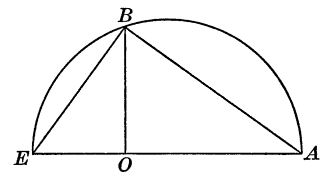


FIG. 2.

But these three planes intersect in the diameter through A of the circumscribed sphere. Extend AO to meet the sphere again in E (Fig. 2). Then EA = diameter of circumsphere $= 2R$. Since A-BCD is a regular tetrahedron, AO is perpendicular to the base, BCD. Hence $\angle AOB$ is a right angle. Also $\angle EBA$, being inscribed in a semicircle, is a right angle. Hence in right $\triangle BOA$,

$$\overline{OA}^2 = \overline{AB}^2 - \overline{BO}^2 = a^2 - \frac{a^2}{3} = \frac{2a^2}{3},\ OA = \frac{a\sqrt{2}}{\sqrt{3}}.$$

In right $\triangle EBA$, $\overline{BA}^2 = EA.\quad OA = 2R\cdot\dfrac{a\sqrt{2}}{\sqrt{3}}$.

$$a^2 = 2R\cdot\frac{a\sqrt{2}}{\sqrt{3}},\qquad R = \frac{a\sqrt{3}}{2\sqrt{2}} = \frac{a\sqrt{6}}{4}.$$

Also solved by R. M. MATHEWS, NATHAN ALTSHILLER, A. M. HARDING, CLIFFORD N. MILLS, WALTER C. EELLS, A. H. HOLMES, HORACE OLSON, J. C. CLAGETT, J. W. CLAWSON.

CALCULUS.

361. Proposed by EMMA M. GIBSON, Drury College.

Determine the system of curves satisfying the differential equation

$$[(1 + x^2)^{1/2} + ny]dx + [(1 + y^2)^{1/2} + nx]dy = 0,$$

and show that the curve which passes through the point $(0, n)$ contains as part of itself the conic

$$x^2 + y^2 + 2xy(1 + n^2)^{1/2} = n^2.$$

(From Forsyth's *Differential Equations*, p. 41.)

SOLUTION BY GEO. W. HARTWELL, Hamline University.

The terms of the given differential equation may be arranged as follows:

$$(1 + x^2)^{\frac{1}{2}}dx + (1 + y^2)^{\frac{1}{2}}dy + nxdy + nydx = 0 \tag{1}$$

and the equation integrated immediately, giving

$$x\sqrt{1 + x^2} + y\sqrt{1 + y^2} + 2nxy + \log(x + \sqrt{1 + x^2})(y + \sqrt{1 + y^2}) = c. \tag{2}$$

The equation of the curve of this system passing through $(0, n)$ is then

$$x\sqrt{1+x^2} + y\sqrt{1+y^2} - n\sqrt{1+n^2} + 2nxy + \log\left[\frac{(x+\sqrt{1+x^2})(y+\sqrt{1+y^2})}{n+\sqrt{1+n^2}}\right] = 0. \tag{3}$$

Solving the equation of the conic for x we have

$$x = -y\sqrt{1+n^2} \pm n\sqrt{1+y^2}.$$

These values of x satisfy (3); hence, the conic must be a part of the curve represented by (3).

362. Proposed by C. N. SCHMALL, New York City.

Having given $y^3 - a^2y + axy - x^3 = 0$, show by Maclaurin's theorem that

$$y = -\frac{x^3}{a^2} - \frac{x^4}{a^3} - \frac{x^5}{a^4} - \cdots,$$

SOLUTION BY A. M. HARDING, University of Arkansas.

We obtain by successive differentiation

$$(3y^2 - a^2 + ax)\frac{dy}{dx} - 3x^2 + ay = 0,$$

$$(3y^2 - a^2 + ax)\frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)^2 + 2a\frac{dy}{dx} - 6x = 0,$$

$$(3y^2 - a^2 + ax)\frac{d^3y}{dx^3} + 18y\frac{d^2y}{dx^2}\frac{dy}{dx} + 6\left(\frac{dy}{dx}\right)^3 + 3a\frac{d^2y}{dx^2} - 6 = 0,$$

$$(3y^2 - a^2 + ax)\frac{d^4y}{dx^4} + 24y\frac{d^3y}{dx^3}\frac{dy}{dx} + 18y\left(\frac{d^2y}{dx^2}\right)^2 + 36\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^2 + 4a\frac{d^3y}{dx^3} = 0,$$

$$(3y^2 - a^2 + ax)\frac{d^5y}{dx^5} + 30y\frac{d^4y}{dx^4}\frac{dy}{dx} + 60y\frac{d^3y}{dx^3}\frac{d^2y}{dx^2} + 60\frac{d^3y}{dx^3}\left(\frac{dy}{dx}\right)^2 + 90\left(\frac{d^2y}{dx^2}\right)^2\frac{dy}{dx} + 5a\frac{d^4y}{dx^4} = 0.$$

When $x = 0$, $y = 0$, a, or $-a$. Choosing the first value of y, we obtain

$$\frac{dy}{dx} = 0, \quad \frac{d^2y}{dx^2} = 0, \quad \frac{d^3y}{dx^3} = -\frac{6}{a^2}, \quad \frac{d^4y}{dx^4} = -\frac{24}{a^3}, \quad \frac{d^5y}{dx^5} = -\frac{120}{a^4}.$$

Hence,

$$y = -\frac{6}{a^2}\frac{x^3}{3!} - \frac{24}{a^3}\frac{x^4}{4!} - \frac{120}{a^4}\frac{x^5}{5!} - \cdots$$

$$= -\frac{x^3}{a^2} - \frac{x^4}{a^3} - \frac{x^5}{a^4} - \cdots.$$

Also solved by PAUL CAPRON and GEO. W. HARTWELL.

NUMBER THEORY.

215. Proposed by R. D. CARMICHAEL, Indiana University.

Find one or more values of n such that a polygon of n sides shall have the number of its diagonals equal to the cube of an integer.

SOLUTION BY WALTER C. EELLS, U. S. Naval Academy.

Since the number of diagonals of an n-sided polygon is $\frac{n(n-3)}{2}$, this is equivalent to the problem: find solutions in integers of $n^2 - 3n = 2k^3$. By the aid of a table of squares it is easily found that the only values of n less than 1,000 satisfying this equation are 9 and 128, for which $k = 3$ and 20 respectively, the number of diagonals being 27 and 8,000.

216. Proposed by ELIJAH SWIFT, University of Vermont.

If p is a prime > 3, show that $\Sigma_{a=1}^{a=p-1} 1/a \equiv 0 \pmod{p^2}$, where $1/a \equiv x$, if $ax \equiv 1 \pmod{p^2}$.

SOLUTION BY THE PROPOSER.

Referring to my solution[1] of algebra problem number 385, I proved that $A_{p-2} \equiv 0 \pmod{p^2}$, where $A_{p-2} \equiv 1\cdot 2\cdot 3 \cdots p-2 + 1\cdot 3\cdot 4 \cdots p-1 + \cdots$.

Hence,

$$A_{p-2} \equiv (1\cdot 2\cdot 3 \cdots (p-1)) \sum_{a=1}^{p-1} \frac{1}{a}.$$

Suppose that

$$1\cdot 2\cdot 3 \cdots (p-1) \equiv -1 + A\cdot p \pmod{p^2}.$$

Then

$$A_{p-2} \equiv \sum_{a=1}^{p-1} \frac{-1 + Ap}{a} = \sum_{a=1}^{p-1} \frac{-1}{a} + pA \sum_{a=1}^{p-1} \frac{1}{a}.$$

But

$$\sum_{a=1}^{a=p-1} \frac{1}{a} \equiv \sum_{a=1}^{a=p-1} a \pmod{p} \equiv 0,$$

since

$$A_{p-2} \equiv 0 \pmod{p^2} \sum_{a=1}^{a=p-1} \frac{1}{a} \equiv 0 \pmod{p^2}.$$

NOTES AND NEWS.

EDITED BY W. DEW. CAIRNS.

Miss MARIE GUGEL, formerly teacher of mathematics in the Toledo, Ohio, high school, is now supervisor of high schools in Columbus. She is secretary of the mathematics section of the Central Association of Science and Mathematics Teachers.

Mr. FORREST R. BAKER, assistant in mathematics at the University of Michigan, died December 6, following an operation for appendicitis.

[1] Volume XXI, page 157, May, 1914.

Professor L. C. KARPINSKI has been promoted to a junior professorship of mathematics in the University of Michigan.

Mr. F. A. FORAKER, of the University of Pittsburgh, contributes to *Education* for December, 1914, an article on "The relation of the symbols of mathematics to the elements of the problems."

At the University of Oklahoma Associate Professor F. C. KENT has resigned and Mr. H. B. GOSSARD, of the Johns Hopkins University, has been appointed to an instructorship in mathematics.

The *Bollettino di bibliografia e storia delle scienza matematiche* for the last quarter of 1914 contains an Italian translation of Professor E. J. WILCZYNSKI's paper on "Some general aspects of modern geometry" which appeared in the *Bulletin of the American Mathematical Society* for April, 1913. The translation is preceded by a brief appreciative account of Professor Wilczynski's fundamental work on projective differential geometry.

An article in the October, 1914, number of the *Bulletin* of the Society for the Promotion of Engineering Education, entitled "What a technical education costs," gives the average annual expense of 65 students at the Massachusetts Institute of Technology. This amounted to \$616.39 for students from a distance and \$327.65 for students living at home.

School Science and Mathematics for November and December, 1914, printed a paper on "Some observations on the study and teaching of mathematics in Germany," read by Professor GORDON N. ARMSTRONG, of Ohio Wesleyan University, before the April meeting of the Ohio Teachers of Mathematics and Science.

In the *Mathematical Gazette* for October, 1914, Mr. W. H. MACAULAY presents the results of an investigation of the problem of dissecting two given rectilineal figures of equal areas, by straight lines, so that the parts of either will fit on the other. The simplest case is that in which the triangles ABC and $A'B'C'$ have a pair of equal sides, AB and $A'B'$, and do not differ too much in shape. They can be divided by a three-part and by a four-part dissection, the former being accomplished by joining E and D, the mid points of AC and BC, to F in AB by lines equal to half of $B'C'$ and $A'C'$ respectively; and analogous lines in triangle $A'B'C'$. With the aid of these two fundamental cases Mr. Macaulay establishes other cases, such as the dissection of any given pair of triangles of equal areas (not differing too much in shape) by a four-part and by a seven-part dissection; of a rectangle and square of equal areas; of a quadrilateral and parallelogram of equal areas, etc.

A recent letter from PROFESSOR CAJORI includes the following interesting notes:

A "Junior Encyclopedia Britannica" is now in preparation. The mathematical articles will be prepared by PHILIP E. B. JOURDAIN, of Girton, near Cambridge. Only the first two volumes will appear until the war is over, but these will contain the two long articles "Arithmetic" and "Algebra," as well as several shorter articles. Jourdain's idea is to treat every mathematical subject historically. For example, instead of attempting to define algebra, which word has meant different things at different times, the plan is to show that algebra grew out of such and such problems and took on such and such meanings, and to give an idea also of the modern, advanced work.

Mr. Jourdain has been working for some time, on a "History of Mathematical Thought," to be published by George Bell, of London. It will contain a very thorough treatment of the development of the leading conceptions of mathematics, such as limit, continuity, etc. The book will carry the subjects down to modern times and will not go into great details about purely technical advances. Somewhat new will be the stress laid on the influence of Zeno on the form which Greek mathematical thought took, also the unconscious and illogical way in which negative, irrational and imaginary numbers were introduced. A good deal of space will be devoted to the modern work and the principles of mathematics.

The Open Court Publishing Co. will shortly bring out two books translated and annotated by Mr. Jourdain. One is a supplementary volume to the English Mach's *Mechanics*, giving the additions made by Mach to the latest German edition. The other book is a translation of Georg Cantor's papers on transfinite numbers in volumes XLVI and XLIX of the *Mathematische Annalen.*

Periodico di Matematica per l'insegnamento secondario, edited by Professor Giulio Lazzeri, is published bi-monthly at Livorno, Italy, and the *Supplemento al Periodico di Matematica*, under the same editorship, appears monthly from November to July inclusive. The former journal is now in the thirtieth year of publication, and the *Supplemento* in the eighteenth year.

The *Periodico* is intended to meet the scholarly demands of the teachers of mathematics in the secondary schools, and also the needs of university students of mathematics and physics. The *Supplemento* is more elementary in character, and aims to excite even in students in the secondary schools a love for the study of mathematics and physics. A somewhat unique feature of the latter journal is a continuous series of prize problems, the competitors being any students of mathematics and physics in the secondary schools, and the prizes being books on mathematics.

The November, 1914, issue (48 pages) of the *Periodico* contains the following articles, all in Italian: C. MINEO, "On the concept of a real number and upon an elementary theorem concerning such numbers"; D. KRYJANOVSKY, "Upon maxima and minima of plane figures (continuation)"; G. LAZZERI, "Static moments, moments of inertia, and moments of higher order (continuation)"; E. PICCIOLI, "The second hypersphere of Lemoine, etc."; G. CANDIDO, "The

solution of the equation

$$\sqrt[2k+1]{A} + \sqrt[2k+1]{B} + \sqrt[2k+]{C} = 0,\text{''}$$

and "On the equation, $\sum_1^k x_i^2 = y^p$". There also appears a review (less than a page) of the new 5-place logarithmic tables, 56 pages, by Professor E. MOUZIN, published by the Fratelli Bocca in Rome.

The November, 1914, issue of the *Supplemento* contains the following articles: S. CATANAIA, "On the solution of literal irrational equations"; G. ASCOLI, "Note on elementary geometry". There is also discussion of a number of problems. The 134th prize problem proposed by G. Ascoli is as follows:

Determine upon the side BC of a triangle ABC a point U which is such that given U' and U'' the projections of U upon AC and AB respectively, the right lines AU, BU', CU'', shall concur in a point. Demonstrate that if V and W are points on CA and AB, defined in an analogous manner, the right lines AU, BV, CW concur in a point.

The initial number of the *Proceedings of the National Academy of Sciences* contains three mathematical articles, as follows: "Recent progress in the theories of modular and formal invariants and in modular geometry," by L. E. DICKSON; "The synthesis of triad systems," by H. S. WHITE; "The ϕ-subgroup of a group of finite order," by G. A. MILLER. Professor E. H. MOORE, University of Chicago, is the mathematical editor of this journal, which is expected to appear monthly, the first issue being that for January, 1915.

It is announced that a joint session of the American Mathematical Society, the American Astronomical Society and Section A (Astronomy and Mathematics) of the American Association for the Advancement of Science will be held at the University of California on Tuesday, August 3. Addresses will be made by Professor C. J. KEYSER, of Columbia University, on "The human significance of mathematics," and by Dr. G. E. HALE, of Mount Wilson Solar Observatory, on "The work of a modern observatory."

A monument to Professor ENRICO BETTI has been erected in the suburban cemetery at Florence. The monument was designed by Professor Ristori of the Accademia di Belle Arti of Florence, and the bronze work was made under the direction of the sculptor and architect, Professor Arcangeli of Florence.

The address by Dr. FRANK SCHLESINGER, of the University of Pittsburgh, as vice-president and chairman of Section A of the American Association for the Advancement of Science at Philadelphia, appears in *Science* for January 22, the subject being "The object of astronomical and mathematical research."

School Science and Mathematics for February prints an address on "Mathematics and life—the vitalizing of secondary mathematics," delivered by Professor R. D. CARMICHAEL before the Kansas Association of Mathematics Teachers in November, together with the attendant discussion by W. T. STRATTON, of Kansas State Agricultural College.

The *Mid-West Quarterly*, a journal owned and controlled by the University of Nebraska, contains an article in the October, 1914, issue, by Professor E. W. DAVIS, on "Charles Peirce at Johns Hopkins." He was the son of the Harvard mathematician, Benjamin Peirce, and was said by Professor Sylvester to have been "a far greater mathematician than his father," though his activities in other fields were many and varied, especially in the domain of logic.

A lecture by Professor E. W. HOBSON on "John Napier and the Invention of Logarithms" has been published by the Cambridge University Press.

An article by L. C. KARPINSKI reprinted from the *Journal of Education*, Boston, on the tests by Cliff W. Stone and S. A. Courtis for determining standards of efficiency in the fundamental operations of arithmetic, is of interest in the line of recent critical examinations of the schools with a view to some definite and precise knowledge of their achievements and shortcomings.

In the September–October number of *Rendiconti del Circolo Matematico di Palermo* appears a paper on "Some properties of closed continuous curves," by Professor ARNOLD EMCH. The November–December number contains a paper on "Birational transformations of the cubic variety in four-dimensional space," by Professor VIRGIL SNYDER, and one on "Algebraic and transcendental numbers," by Professor G. N. BAUER and Dr. H. L. SLOBIN.

School and Society is the name of a new weekly journal edited by Professor J. MCKEEN CATTELL and printed by the Science Press at Lancaster, Pa. The scope of this journal is broad. To quote from the prospectus: "The journal will follow the general lines that have made *Science* of service in the sciences, coöperating with publications in special fields, aiming to become the professional journal for those engaged in the work of our lower and higher schools, and to be of interest to the wider public for whom education is of vital concern. It will emphasize the relations of education to the social order, scientific research in education and its applications, freedom of discussion, and reports and news of events of educational interest."

The committee of the New England Association of Mathematics Teachers appointed to consider the status of secondary mathematics made a preliminary report at the meeting of December 5, 1914, at which time addresses were made by Superintendent MORRISON, of New Hampshire, and Commissioner SNEDDEN, of Massachusetts. The committee has had extended correspondence with Commissioner Snedden which is likely to be published in the near future.

The committee has recently organized subcommittees on: The status of algebra; The practicability of an introductory composite course; The training of teachers of mathematics; School programs; Psychological aspects of the subject; and Mathematics for girls. A syllabus for each topic is in preparation, and correspondence with persons interested would be welcome. See Question 24 under QUESTIONS and DISCUSSIONS in this issue.

The prospectus of *The University of Chicago Science Series* states that: "The volumes of the series will differ from the discussions generally appearing in technical journals in that they will present the complete results of an experiment or series of investigations which have previously appeared only in scattered articles, if published at all. On the other hand, they will differ from detailed treatises by confining themselves to specific problems of current interest and in presenting the subject in as summary a manner and with as little technical detail as is consistent with sound method. They will be written not only for the specialist but also for the educated layman. The size of the volumes will range from fifty to one hundred and fifty pages."

The mathematical publications thus far planned or in preparation are: "Finite collineation groups," by Professor HANS BLICHFELDT, of Stanford University; and "Linear integral equations in general analysis," by Professor E. H. MOORE, of the University of Chicago. The editors of this series are Professors E. H. MOORE, J. M. COULTER, and R. A. MILLIKAN.

The meeting of the Michigan Schoolmasters' Club will be held at Ann Arbor, April 1, 2, and 3, 1915. The program of the mathematics section of the Club will consist entirely of short discussions on practical phases of the teaching of high school mathematics. On Thursday, April first, the teachers will meet at a luncheon at Newberry Hall, Ann Arbor, and the papers will be presented at the same place. The discussion on Thursday will center about the two topics: Practical applications of high school mathematics, and Correlation between mathematics and other branches and correlation of the various mathematical disciplines. Papers will be presented on the correlation between arithmetic and algebra, between algebra and geometry, and between mathematics and physics. On Friday the discussion will center about the two topics: The teaching of algebra, and The teaching of geometry.

A letter from Professor Bôcher calls attention to the fact that the holders of the Benjamin Peirce instructorships at Harvard University may not "pursue courses for higher degrees" as stated in the February number of the MONTHLY, but that as a rule the incumbents will hold the degree of doctor of philosophy. He states that the appointees for 1915–16 are Dr. EDWARD KIRCHER and Dr. G. A. PFEIFFER, both of whom received the degree of Ph.D. in 1914, the former at the University of Illinois, and the latter at Columbia University.

Mr. Bradbury, whose death was announced in the February issue, was head master of the Cambridge Latin School and not of the Boston Latin School, as stated.

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THE HISTORY OF ZENO'S ARGUMENTS ON MOTION:

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

V.

By FLORIAN CAJORI, Colorado College.

5. BAYLE, LEIBNIZ AND OTHER CONTINENTAL WRITERS.

Speculations by Pierre Bayle. An unusually elaborate, detailed, and critical discussion of Zeno's arguments and Aristotle's refutations is given by Pierre Bayle in the article "Zenon d'Elée," printed in his *Dictionnaire historique et critique*, 1696. Our quotations are from the English translation of the dictionary, brought out in 1710 in London. Bayle was a noted French skeptical philosopher; his article on Zeno has been widely quoted. He begins with observations on the "Arrow." Every one admits, he says, that two bodies cannot be in the same place at the same time; that "two parts of Time cannot exist together" is a theorem which "requires a little more reflection in order to apprehend it." Bayle continues:

"I will render it more obvious by an Instance. I say then that what suits Monday and Tuesday with respect to succession, suits every portion of Time whatsoever. Since then it is impossible for Monday and Tuesday to exist together, and that of necessity Monday must cease to be before Tuesday begins to be, there is no part of time whatsoever, which can co-exist with another; each must exist alone; each must begin to be, when the precedent ceaseth to be; and each must cease to be before the following can begin to exist. From whence it follows, that Time is not divisible *in infinitum*, and that the successive duration of things is composed of Moments, properly so called, each of which is simple and indivisible, perfectly distinct from time past and future, and contains no more than the present time. Those who deny this Consequence, must be given up to their Stupidity, or their want of Sincerity, or the insurmountable power of their prejudices. But if you once grant that the present time is indivisible, you will be unavoidably obliged to admit Zeno's Objection. You cannot find an instant when the Arrow leaves its place; for if you find one, it will be at the same time in that place, and yet not there. Aristotle contents himself with answering, that Zeno very falsely supposes the indivisibility of Moments."

This is Bayle's singular argument by which he tries to show that time is composed of a finite number of indivisible parts, and that, in consequence of this property of time, the difficulties of Zeno's argument that the arrow does not

move, are satisfactorily removed. The heat displayed by Bayle at the close leads one to surmise that he himself had encountered opposition to his own explanation.

While Bayle denied the infinite divisibility of time, he admitted the infinite divisibility of space. He gives Zeno's argument in the "Dichotomy" and then says against Aristotle:

"To this Aristotle makes a wretched Answer: He sayth that a foot of matter being no otherwise infinite than in Power, may very well be run through in a finite time. . . . You have here two Particulars: 1. That each part of Time is divisible *in infinitum;* which is invincibly refuted above. 2. That a Body is only Infinite in Power. Which signifies that the Infinity of a Foot of Matter, consists in that it may be divided without end into smaller parts, but not in its being actually susceptible of that division. To urge this is to impose on the World; for if Matter is divisible *in infinitum,* it actually contains an infinite number of parts, and is not therefore an infinite in Power, but an Infinite which really and actually exists. . . . Don't Aristotle and his Followers assert, that an Hour contains an infinity of parts? Wherefore when it is past, it must be owned that an infinity of parts did actually exist one after another. Is this a virtual, and not an actual infinity? Let us then say that this Distinction is null, and that Zeno's Observation remains in full force. . . . Let us content our selves in this place with observing, that the subterfuge of the Infinity of the Parts of Time is null; for if there were in an Hour an infinity of Parts, it could never either begin or end."

What Bayle says in criticism of Aristotle has considerable force; what he says by way of a constructive argument—accepting the infinite divisibility of space, but denying it for time—fails to convince. Bayle says but little on "Achilles" but makes extensive, though uninteresting, remarks on the "Stade."

After this preliminary survey of Zeno and Aristotle, Bayle starts out anew, stating that Zeno very probably alleged other arguments which were perhaps the same as those he himself was about to mention, "some of which oppose the existence of Extension, and seem much stronger than all the Reasons which the Cartesians can allege." He continues:

"I am apt to think that those who would revive Zeno's Opinion, ought to argue thus. 1. There is no Extension, therefore there is no Motion. The Consequence is good, for what hath no Extension takes up no room, and what takes up no room cannot possibly pass from one place to another, nor consequently move. This is incontestable: The difficulty is then to prove that there is no Extension. Zeno might have argued thus. Extension cannot be composed of Mathematical Points, Atoms, or Parts divisible *in infinitum;* therefore its Existence is impossible. . . . A few Words shall suffice as to Mathematical Points; for . . . several nullities of Extension joined together will never make an Extension. Consult the first Body of Scholastical Philosophy that comes to hand, and you will find the most convincing Reasons in the World, supported by many Geometrical Demonstrations against the existence of these Points."

After this appeal to scholastic philosophy on the impossibility of a continuum made up of points, Bayle argues against the extended indivisible particles, called Epicurean atoms; their indivisibility is "chimerical," as each atom has a right side and a left side. Divisibility *in infinitum* leads him to the following caustic observations:

"The divisibility to *in infinitum* is an Hypothesis embraced by Aristotle, and almost all the Professors of Philosophy, in all Universities for several Ages. Not that they comprehend it, or can answer the Objections it is liable to; but because having clearly apprehended the impossibility of either Mathematical or Physical Points, they found no other side but this to take. Besides, this Hypothesis affords great Conveniences: For when their Distinctions are exhausted, without being able to render this Doctrine comprehensible, they shelter themselves in the nature of the Subject, and allege, that our Understandings being limited, none ought to be surprized that they

cannot resolve what relates to Infinity, and that it is essential to such a Continuity to be liable to such Difficulties as are insurmountable by Humane Nature. . . . To be convinced of their weakness, it is enough to remember that the strongest of them [the three hypotheses, points, atoms, parts infinitely divisible], that which best disputes the ground, is the Hypothesis of the divisibility to *in infinitum.* The Schoolmen have armed it from head to foot with all the Distinctions which their great leisure would allow them to invent: But all this only serves to afford their Scholars Matter for talk upon a public Disputation, that their Relations may not suffer the Disgrace of seeing them mute. And accordingly a Father or a Brother go away better satisfy'd, when the Scholar distinguishes betwixt a *Categorematical Infinite*, and a *Syncategorematical* one . . . than if he should answer nothing. It was therefore necessary for the Professors to invent a sorte of Jargon; but all the pains which they have taken, will never be able to obscure this Notion which is as clear and evident as the Sun: *An infinite number of Parts of Extension, each of which is extended, and distinct from all others, as well with respect to its Entity, as the room is taken up, cannot be contained in a Space One hundred Millions of times less than the Hundred Thousandth part of a Barly Corn.*"

That Bayle was thinking more of external realities than pure concepts of the human mind appears from the following quotation:

"What the Mathematicians acknowledge with respect to Lines and Superficies, with which they demonstrate so many excellent things, must be owned to be true of Bodies. They honestly own that length and breadth without depth, are things which cannot possibly exist anywhere but in our Imagination. As much may be said of the three Dimensions. They cannot find any room any where besides in our Minds; nor can they exist any other way than Notionally. . . ."

The demonstrations which have been given to show infinite divisibility are interpreted by Bayle as really proving that extension does not exist.

"In the 1. place I observe that some of these Demonstrations are made use of against those who affirm that Matter is composed of Mathematical Points. It is objected to them, that the Sides of a Square would be equal to the Diagonal, and that amongst concentrical Circles, the least would be equal to the largest. This Consequence is proved by making it appear that the right Lines which may be drawn from one of the sides of a Square to another will fill the Diagonal, and that all the right Lines which may be drawn from the Circumference of the largest Circle, will find room in the smallest Circumference. . . . In the 2. place I affirm that it being very true that if Circles did exist, as many right lines might be drawn from the Circumference, to the Center, as there are parts in the Circumference, it follows that the existence of a Circle is impossible. I assure myself that it will be allowed me that every Being which cannot exist, without containing properties which cannot exist, is impossible: But a round Extension cannot exist, without having a Center, in which as many right Lines as there are parts in the Circumference meet; and it is certain that such a Center cannot exist; It must then be owned that the Existence of this round Extension is impossible."

Further indication of the difficulties encountered in the effort to construct a non-contradictory continuum is exhibited in the following passage by Bayle:

". . . a Body in motion, rolling in a sloping Table, could never fall off the said Table; for before it falls, it must of necessity touch the last part of the Table. And how will it touch that, since all those parts which you will take for the last, contain an Infinity of parts, and an infinite Number hath no part which can be last? This Objection obliged some Scholastic Philosophers to suppose that Nature hath intermixed Mathematical Points with the parts divisible *in infinitum*, to the end that they may serve to connect them, and compose the extremities of Bodies."

And in conclusion Bayle says:

"Thus . . . we may suppose our Zeno of Elea to have opposed Motion. I will not aver that his Reasons persuaded him that nothing moved. . . . If I should judge of him by my self, I should affirm that he as well as others believed the motion of Extension; for tho' I find my self very incapable of solving all the difficulties which we have just now seen . . . I am persuaded that the exposition of these Arguments may be of great use with respect to Religion. . . . The advantage which may be drawn from these speculations is not merely to acquire this sort of Knowledge, which in itself is very barren; but to learn to know the bounds of our understanding."

Bayle tells also of the Sophist Diodorus who lectured against the existence of motion.[1] Having put his shoulder out of joint, he went to a physician to have it set. How? said the Doctor. Your Shoulder dislocated! That cannot be; for, if it moved, it did so either in the place where it was, or in the place where it was not. But it did not move either in the place where it was, or in the place where it was not, for it could neither act nor suffer in the place where it was not.

Bayle's article, though by far the fullest discussion of Zeno given up to that time, is not an illuminating contribution. It is prepared without much co-ordination. At first he refers approvingly to the infinite divisibility of space, later he speaks sneeringly of the advocates of that view. No mention is made of the important work of Gregory St. Vincent. Bayle's general attitude in his article is that of a skeptic.

Views of Leibniz. About the time when Bayle prepared his article, Leibniz touched upon the "Achilles" in his correspondence with the French philosopher Foucher. We may premise that lack of space prevents us from attempting a systematic exposition of Leibniz's infinitesimals and their use in his calculus. Like Newton, Leibniz changed his point of view on some fundamental concepts of mathematics, as the years rolled on. This change of base cannot be brought as a charge against either of them. They were encountering most subtle problems, in many different fields of inquiry; their ideas were in a state of flux. It is owing to this circumstance, as Vivanti has pointed out,[2] that different authors have attributed to Leibniz opposite views and each was able to fortify his contention by direct quotation. Thus Wolf, Achard, Gerdil, and Mansion denied that Leibniz admitted the existence of infinitesimals different from zero; Grandi claimed the opposite; Cohen and Lasswitz attributed to him the concept of the intensive infinitesimal, that is, an infinitesimal considered as the generator of finite magnitude, though itself without magnitude. In March, 1693, Foucher wrote Leibniz a letter in which he asks for information, how Leibniz could consistently admit divisibles and also indivisibles, and dwells upon the difficulties offered by the various alternatives: indivisible instants corresponding to divisible dots (points), or divisible instants corresponding to indivisible dots, or divisible instants corresponding to divisible dots. In the case of the last alternative, "on ne pourra resoudre la difficulté des Sceptiques, ni montrer comment Achille doit aller plus vite qu'une tortuë."[3] To this Leibniz replies[4] that twenty years previously he had written two discourses on motion which may contain some things of value but which contain passages on which he considers himself now better informed, "et entre autres, je m'explique tout autrement aujourdhuy

[1] Sextus Empiricus, lib. 2, c. 22.

[2] Giulio Vivanti, "Il concetto d'infinitesimo e la sua applicazione alla matematica. Saggio storico," *Giornale di matematiche di Battaglini,* Vol. 38 e 39, Estratto, Napoli, 1901, p. 11. This research is a most valuable one, containing extensive quotations from a large number of original sources many of which are difficult of access.

[3] *Die philosophischen Schriften von Gottfried Wilhelm Leibniz,* herausgegeben v. C. I. Gerhardt, Bd. I, Berlin, 1875, p. 411.

[4] *Loc. cit.,* p. 415.

sur les indivisibles. C'estoit l'essay d'un jeune homme qui n'avoit pas ancor approfondi les mathématiques." Later passages in this letter are of interest in the light of recent concepts of the atom, as well as in the light of the modern theory of the continuum:

"Quand aux *indivisibles*, lorsqu'ou entend por là les simples extremités du temps ou de la ligne, ou n'y sçauroit concevoir de nouvelles extremités ny des parties actuelles ny potentielles. Ainsi les points ne sont ny gros ny petits, et il ne faut point de saut pour les passer. Cependant le continu, quoy qu'il ait partout de tels indivisibles, n'en est point composé, comme il semble que les objections de Sceptiques le supposent, qui, à mon avis, n'ont rien d'insurmontable, comme on trouvera en les redigeant en forme. Le père Gregoire de S. Vincent a fort bien montré par le calcul même de la divisibilité à infini, l'endroit où Achille doit attraper la tortue qui le devance, selon la proportion des vitesses. Ainsi la Geometrie sert à dissiper ces difficultés apparentes. Je suis tellement pour l'*infini actuel*, qu'au lieu d'admettre que la nature l'abhorre, comme l'on dit, vulgairement, je tiens qu'elle l'affecte partout, pour mieux marquer les perfections de son auteur. Ainsi je crois qu'il n'y a aucune partie de la matiere qui ne soit, je ne dis pas divisible, mais actuellement divisée, et par consequent, la moindre particelle doit estre considerée comme monde plein d'une infinité de creatures differentes."

It is interesting to observe that Leibniz's comment on Gregory St. Vincent's explanation of the "Achilles" was favorable, as were all comments of that time with which we are familiar. It was a time when a circle was quite generally looked upon as a polygon with an infinite number of sides; hence, no ultra refinements should be expected. Worthy of notice is a passage in a letter of Leibniz[1] to John Bernoulli I, dated Aug. 22, 1698, in which Leibniz declares that Burcher de Volder, and Gregory St. Vincent before him, rejected the axiom that the whole is greater than its part when it applied to infinity. Volder was a man of prominence in the Netherlands, as appears from the fact that he was selected to edit the works of Huygens. Leibniz could not agree with the Dutch scientists and called their views absurd. It is interesting to see how this idea of the whole not being greater than certain of its parts, so clearly brought out by Galileo for infinite aggregates, every now and then forced itself upon the attention of men pondering on the subject of infinity. The Spaniard, Juan Andrès,[2] quotes with approval from Christian Wolff's widely used book, *Elementa matheseos universae* (Arth. num. 86), a proof of the theorem that "the whole is greater than its part." On the other hand, the opposite view, held in the seventeenth century (as we have seen) by Galileo and Volder, found utterance in the eighteenth century in a book by Johann Schultz.[3]

Some Eighteenth Century Discussion. While considerable discussion took place on Zeno's arguments in the seventeenth century, by writers like Biancani, Gregory St. Vincent, Peter Bayle, Descartes, and Leibniz, comparatively little was said on this subject during the eighteenth century. There was tremendous activity during the eighteenth century in the fuller development of the differential and integral calculus and its applications. With it came a more systematic

[1] Got. Gul. Leibnitii et Johan. Bernoullii *Commercium Philosophicum et mathematicum*, T. 1, Lausannæ et Genevæ, 1745, pp. 389, 397.

[2] Juan Andrès, *De studiis philosophicis et matematicis*, Matriti, 1789.

[3] J. Schultz, *Versuch einer genauen Theorie des Unendlichen*, Königsberg und Leipzig, 1778, p. 87.

development of the theory of limits, but that development was not such as to really throw much new light upon infinite divisibility or the ability of variables to reach their limits. Much might be said on discussions of the infinite, but we shall confine our attention to views that bear more directly upon the subject of our inquiry.

The Italian philosopher and philologist, Jacopo Facciolati, of Padua, wrote upon the "Achilles" in *Acroases dialecticae*, Venetiis, 1750. As reported by Hoffbauer,[1] Facciolati makes an assumption, in accordance with which the tortoise is never caught, though Zeno's alleged contention that the swifter cannot catch the slower is not established thereby. Suppose a, b, c, d, . . . are points on a line, such that the distance bc is one-tenth the preceding distance ab, etc. The extra assumption was to the effect that both Achilles and the tortoise made stops at the points a, b, c, . . . so that the time of transit from one letter to the next did not fall below a certain minimum. This solution of the puzzle can hardly be ranked as a real advance. More searching was Father Gerdil (1718–1802) of Turin who ranked high as a professor and philosopher, and was finally given a cardinal's hat. In his article, *De l'infini absolu consideré dans la Grandeur*,[2] he quotes from an article by the French professor of mathematics and philosophy, l'abbé Deidier (1696–about 1746), who said that Zeno's conclusion is absurd, except on two suppositions: the first is that Achilles took an infinite number of steps to cover the first league, in which case he never reached his goal; the second is that when he passed $\frac{1}{10}$ of the previous distance, his steps also became ten times shorter, so that he could not reach the tortoise. As both of these suppositions are ridiculous and impossible, it follows that Zeno's argument is a mere sophism. If some one objects by saying that Achilles must travel $\frac{1}{9}$ of a league, which he cannot do since he has to pass through an infinite progression $\frac{1}{10}$, $\frac{1}{100}$, . . . , I reply that this is a sophism as simple as the first, for Achilles continually travels at a uniform rate.

Father Gerdil endorses abbé Deidier's views. He himself points out that, if the tortoise has at starting the lead of 1 league, it travels a distance x before it is caught, where x is determined by $10x = 1 + x$. His mode of solving the "Achilles" consists in avoiding the summation of an infinite progression by addition of its terms, and in determining by one stroke the value represented by that progression. An infinite progression has no last term, yet says he, the number of terms does not constitute an actual infinity. His argument against the possibility of an actual infinity carried great weight with Cauchy.[3]

Passing to Germany we meet first with a philosophical publication by Johann

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[1] J. S. Ersch und J. G. Gruber, Allg. Encyclopädie der Wissensch. u. Künste, Leipzig, 1818, Art. "Achilles."

[2] *Mélanges de philosophie et de mathématique de la Société Royale de Turin*, 1760–1761, Suppl., p. 1. Georg Cantor gives also the following article by Gerdil: "Essai d'une démonstration mathématique contre l'existence éternelle de la matière et du mouvement, déduite de l'impossibilité demontrée d'une suite actuellement infinie de termes, soit permanents, soit successifs," *Opere edite et inedite del cardinale Giacinto Sigismondo Gerdil*, T. IV, p. 261, Rome, 1806.

[3] See Georg Cantor, "Ueber die verschiedenen Standpunkte in Bezug auf das actuale Unendliche," *Zeitsch. f. Philos. u. Philos. Kritik*, Bd. 88, p. 224.

Gottlieb Waldin,[1] professor of mathematics at Marburg, who declares Zeno's proofs invalid, because Zeno assumes the existence of motion, the very thing in dispute.

THE IDENTICAL RELATIONS BETWEEN THE DIRECTION COSINES OF ONE OBLIQUE COORDINATE SYSTEM REFERRED TO ANOTHER OBLIQUE SYSTEM.

By HENRY D. THOMPSON.

I. *Introduction.*

If $Oxyz$, $Ox'y'z'$ are any two oblique coordinate systems, then connecting the cosines of the fifteen angles between the six lines there are identical relations of the fourth degree which are simple and which are the exact counterpart of the well known twenty-two relations between two orthogonal systems. Relations for the oblique case have been given; for example, by Grunert[2] and by Sturm,[3] but some of these are of a degree higher than the fourth, and they are not always so easy to use as the orthogonal relations. In the *American Journal of Mathematics*, Vol. XXXV, p. 427, it has been proved that the so called Lamé and Gauss equations in the theory of surfaces are special cases of equations holding for oblique triple systems of surfaces. In the same way, in all the cases tried, it has been found that, by the use of the identical relation between two sets of four directions in space, all the usual orthogonal relations can be generalized and the corresponding formulas for the oblique cases can be obtained. This method will be employed here to obtain also the simple relations between oblique coordinate systems.

Let $Ox_0y_0z_0$ be an orthogonal system. Define the (direction) cosines of the angles between the various lines of $Oxyz$, $Ox'y'z'$, $Ox_0y_0z_0$ by the following schemes, $c_{ii} = 1$, $c_{ij} = c_{ji}$, $c_{ii}' = 1$, $c_{ij}' = c_{ji}'$,

$$
\begin{array}{c|ccc}
C & x & y & z \\ \hline
x & c_{11} & c_{12} & c_{13} \\
y & c_{21} & c_{22} & c_{23} \\
z & c_{31} & c_{32} & c_{33}
\end{array}
\qquad
\begin{array}{c|ccc}
C' & x' & y' & z' \\ \hline
x' & c_{11}' & c_{12}' & c_{13}' \\
y' & c_{21}' & c_{22}' & c_{23}' \\
z' & c_{31}' & c_{32}' & c_{33}'
\end{array}
\qquad
\begin{array}{c|ccc}
D & x' & y' & z' \\ \hline
x & l_1 & m_1 & n_1 \\
y & l_2 & m_2 & n_2 \\
z & l_3 & m_3 & n_3
\end{array}
\qquad
\begin{array}{c|ccc}
A & x_0 & y_0 & z_0 \\ \hline
x & \lambda_1 & \mu_1 & \nu_1 \\
y & \lambda_2 & \mu_2 & \nu_2 \\
z & \lambda_3 & \mu_3 & \nu_3
\end{array}
\qquad
\begin{array}{c|ccc}
A' & x_0 & y_0 & z_0, \\ \hline
x' & \lambda_1' & \mu_1' & \nu_1' \\
y' & \lambda_2' & \mu_2' & \nu_2' \\
z' & \lambda_3' & \mu_3' & \nu_3'
\end{array}
$$

Call the corresponding five determinants C, C', D, A, A', and represent each cofactor in these determinants by the capital letter and the subscripts of the corresponding element. Since $\lambda_i^2 + \mu_i^2 + \nu_i^2 = c_{ii}$, $\lambda_i\lambda_j + \mu_i\mu_j + \nu_i\nu_j = c_{ij}$, etc., $i = 1, 2, 3$; $j = 1, 2, 3$; direct multiplication gives that $A^2 = C$, $A'^2 = C'$, and $A \cdot A' = D$, or $D = C^{\frac{1}{2}}C'^{\frac{1}{2}}$.

In the orthogonal case, only the elements of D appear, and in accordance with

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[1] J. G. Waldin, *Erste Gründe der allgemeinen und besondern Vernunftlehre*, Marburg, 1782, p. 26. Our information about Waldin is drawn from a history of Zeno's arguments by Eduard Wellmann, entitled "Zenos Beweise gegen die Bewegung und ihre Widerlegungen," in *Programm des Friedrichs-Gymnasiums zu Frankfurt A. O., für das Schuljahr* 1869–1870. Frankfurt A. O. 1870, p. 14.

[2] *Arch. d. Math.*, 34, p. 142 and fol.

[3] *Arch. d. Math.*, 3te R., 22, p. 327.

that convenient usage, in what follows the identities will be considered as relations in the l_i, m_i, n_i, with the c_{ij}, c_{ij}', C, C', D considered as entering in the coefficients.

II. *The twelve bilinear equations between the elements in the rows and columns of D.*

Express the identical relation[1] between any two sets of four directions in space (d_1, d_2, d_3, d_4; d_1', d_2', d_3', d_4') by the symbol $\left\{\begin{matrix} d_1 & d_2 & d_3 & d_4 \\ d_1' & d_2' & d_3' & d_4' \end{matrix}\right\} = 0$; then the six identical relations

$$\left\{\begin{matrix} x & y & z & x' \\ x & y & z & x' \end{matrix}\right\} = 0, \quad \left\{\begin{matrix} x & y & z & y' \\ x & y & z & y' \end{matrix}\right\} = 0, \quad \left\{\begin{matrix} x & y & z & z' \\ x & y & z & z' \end{matrix}\right\} = 0,$$

$$\left\{\begin{matrix} x' & y' & z' & x \\ x' & y' & z' & x \end{matrix}\right\} = 0, \quad \left\{\begin{matrix} x' & y' & z' & y \\ x' & y' & z' & y \end{matrix}\right\} = 0, \quad \left\{\begin{matrix} x' & y' & z' & z \\ x' & y' & z' & z \end{matrix}\right\} = 0,$$

e panded in terms of l_i, m_i, n_i are

$$\begin{aligned}
C_{11}\, l_1^2 + C_{22}\, l_2^2 + C_{33}\, l_3^2 + 2C_{23}\, l_2\, l_3 + 2C_{31}\, l_3\, l_1 + 2C_{12}\, l_1\, l_2 - C &= 0,\\
C_{11}\, m_1^2 + C_{22}\, m_2^2 + C_{33}\, m_3^2 + 2C_{23}\, m_2 m_3 + 2C_{31}\, m_3 m_1 + 2C_{12}\, m_1 m_2 - C &= 0,\\
C_{11}\, n_1^2 + C_{22}\, n_2^2 + C_{33}\, n_3^2 + 2C_{23}\, n_2\, n_3 + 2C_{31}\, n_3\, n_1 + 2C_{12}\, n_1\, n_2 - C &= 0,\\
C_{11}'\, l_1^2 + C_{22}' m_1^2 + C_{33}'\, n_1^2 + 2C_{23}' m_1\, n_1 + 2C_{31}'\, n_1\, l_1 + 2C_{12}'\, l_1 m_1 - C' &= 0,\\
C_{11}'\, l_2^2 + C_{22}' m_2^2 + C_{33}'\, n_2^2 + 2C_{23}' m_2\, n_2 + 2C_{31}'\, n_2\, l_2 + 2C_{12}'\, l_2 m_2 - C' &= 0,\\
C_{11}'\, l_3^2 + C_{22}' m_3^2 + C_{33}'\, n_3^2 + 2C_{23}' m_3\, n_3 + 2C_{31}'\, n_3\, l_3 + 2C_{12}'\, l_3 m_3 - C' &= 0.
\end{aligned}$$

These six equations are the counterpart of the six equations which state that the sum of the squares of the direction cosines is unity when the axes are rectangular, and were given by Grunert, *l. c.*, p. 142.

In passing also, it is useful to note that when L, M, N are any three numbers proportional to the oblique direction cosines of a line, the oblique direction cosines themselves, l, m, n, can be found. For, take R so that $l = L/R$, $m = M/R$, $n = N/R$; these values in the identical relation give

$$\begin{vmatrix} c_{11} & c_{12} & c_{13} & L \\ c_{21} & c_{22} & c_{23} & M \\ c_{31} & c_{32} & c_{33} & N \\ L & M & N & R^2 \end{vmatrix} = 0; \text{ whence } R = \pm \frac{1}{C^{\frac{1}{2}}} \left\| \begin{matrix} c_{11} & c_{12} & c_{13} & L \\ c_{21} & c_{22} & c_{23} & M \\ c_{31} & c_{32} & c_{33} & N \\ L & M & N & O \end{matrix} \right\|^{\frac{1}{2}}$$

And in the same way, when two lines Od and Od' are given by numbers proportional to the oblique direction cosines, respectively, viz., L, M, N and L', M', N', then

$$\pm \cos (dd') = \begin{vmatrix} c_{11} & c_{12} & c_{13} & L \\ c_{21} & c_{22} & c_{23} & M \\ c_{31} & c_{32} & c_{33} & N \\ L' & M' & N' & O \end{vmatrix} \Bigg/ \left\| \begin{matrix} c_{11} & c_{12} & c_{13} & L \\ c_{21} & c_{22} & c_{23} & M \\ c_{31} & c_{32} & c_{33} & N \\ L & M & N & O \end{matrix} \right\|^{\frac{1}{2}} \left\| \begin{matrix} c_{11} & c_{12} & c_{13} & L' \\ c_{21} & c_{22} & c_{23} & M' \\ c_{31} & c_{32} & c_{33} & N' \\ L' & M' & N' & O \end{matrix} \right\|^{\frac{1}{2}}$$

To obtain the six equations which are the generalized form of the six equations which state that the product of the corresponding direction cosines of two axes of a set are zero when the axes are rectangular, take the six identical relations

$$\left\{\begin{matrix} x & y & z & y' \\ x & y & z & z' \end{matrix}\right\} = 0, \quad \left\{\begin{matrix} x & y & z & z' \\ x & y & z & x' \end{matrix}\right\} = 0, \quad \left\{\begin{matrix} x & y & z & x' \\ x & y & z & y' \end{matrix}\right\} = 0,$$

$$\left\{\begin{matrix} x' & y' & z' & y \\ x' & y' & z' & z \end{matrix}\right\} = 0, \quad \left\{\begin{matrix} x' & y' & z' & z \\ x' & y' & z' & x \end{matrix}\right\} = 0, \quad \left\{\begin{matrix} x' & y' & z' & x \\ x' & y' & z' & y \end{matrix}\right\} = 0.$$

[1] Cf. Pascal-Timerding, Rep. d. h. Geom., 2d Ed., part 1, 1910, p. 74.

These expanded in terms of l_i, m_i, n_i are

$$
\begin{aligned}
&C_{11}\, m_1 n_1 + C_{22}\, m_2 n_2 + C_{33}\, m_3 n_3 + C_{23}\, (m_2 n_3 + m_3 n_2) + C_{31}\, (m_3 n_1 + m_1 n_3)\\
&\qquad\qquad + C_{12}\, (m_1 n_2 + m_2 n_1) - c_{23}'\, C \;= 0,\\
&C_{11}\, n_1\, l_1 + C_{22}\, n_2\, l_2 + C_{33}\, n_3\, l_3 + C_{23}\, (n_2\, l_3 + n_3\, l_2) + C_{31}\, (n_3\, l_1 + n_1\, l_3)\\
&\qquad\qquad + C_{12}\, (n_1\, l_2 + n_2\, l_1) - c_{31}'\, C \;= 0,\\
&C_{11}\, l_1 m_1 + C_{22}\, l_2 m_2 + C_{33}\, l_3 m_3 + C_{23}\, (l_2 m_3 + l_3 m_2) + C_{31}\, (l_3 m_1 + l_1 m_3)\\
&\qquad\qquad + C_{12}\, (l_1 m_2 + l_2 m_1) - c_{12}'\, C \;= 0,\\
&C_{11}'\, l_2\, l_3 + C_{22}' m_2 m_3 + C_{33}'\, n_2 n_3 + C_{23}'(m_2 n_3 + m_3 n_2) + C_{31}'(n_2\, l_3 + n_2\, l_3)\\
&\qquad\qquad + C_{12}'(l_2 m_3 + l_3 m_2) - c_{23}\, C' = 0,\\
&C_{11}'\, l_3\, l_1 + C_{22}' m_3 m_1 + C_{33}'\, n_3 n_1 + C_{23}'(m_3 n_1 + m_1 n_3) + C_{31}'(n_3\, l_1 + n_1\, l_3)\\
&\qquad\qquad + C_{12}'(l_3 m_1 + l_1 m_3) - c_{31}\, C' = 0,\\
&C_{11}'\, l_1\, l_2 + C_{22}' m_1 m_2 + C_{33}'\, n_1 n_2 + C_{23}'(m_1 n_2 + m_2 n_1) + C_{31}'(n_1\, l_2 + n_2\, l_1)\\
&\qquad\qquad + C_{12}'(l_1 m_2 + l_2 m_1) - c_{12}\, C' = 0.
\end{aligned}
$$

The left sides of these last six equations are factors in the equations given by Grunert, *l. c.*, p. 143, etc.

III. *The value of D.*

Since $C^{\frac{1}{2}}$ is the von Staudt triedral sine and is positive or negative according as $Ox_0y_0z_0$ and $Oxyz$ are of the same or opposite senses,[1] and the same for $C'^{\frac{1}{2}}$ with regard to $Ox_0y_0z_0$ and $Ox'y'z'$, therefore D is a positive or negative proper fraction according as $Oxyz$ and $Ox'y'z'$ are of the same or opposite senses. And, for rigid $Oxyz$, and rigid $Ox'y'z'$, D is invariant for all the relative positions. This is the counterpart of the Jacobi (1835) formula, $D = \pm 1$, for orthogonal systems.

IV. *The linear relations between the elements of D and their cofactors.*

The nine identical relations

$$
\left\{\begin{matrix} x' & y' & z' & x \\ x & y & z & x' \end{matrix}\right\} = 0, \quad
\left\{\begin{matrix} x' & y' & z' & y \\ x & y & z & x' \end{matrix}\right\} = 0, \quad
\left\{\begin{matrix} x' & y' & z' & z \\ x & y & z & x' \end{matrix}\right\} = 0,
$$

$$
\left\{\begin{matrix} x' & y' & z' & x \\ x & y & z & y' \end{matrix}\right\} = 0, \quad
\left\{\begin{matrix} x' & y' & z' & y \\ x & y & z & y' \end{matrix}\right\} = 0, \quad
\left\{\begin{matrix} x' & y' & z' & z \\ x & y & z & y' \end{matrix}\right\} = 0,
$$

$$
\left\{\begin{matrix} x' & y' & z' & x \\ x & y & z & z' \end{matrix}\right\} = 0, \quad
\left\{\begin{matrix} x' & y' & z' & y \\ x & y & z & z' \end{matrix}\right\} = 0, \quad
\left\{\begin{matrix} x' & y' & z' & z \\ x & y & z & z' \end{matrix}\right\} = 0,
$$

expanded give ($i = 1, 2, 3$)

$$
\begin{aligned}
D\, l_1 &= \Sigma c_{i1}(c_{11}'L_i + c_{12}'M_i + c_{13}'N_i), && (1)\\
D\, l_2 &= \Sigma c_{i2}(c_{11}'L_i + c_{12}'M_i + c_{13}'N_i), && (2)\\
D\, l_3 &= \Sigma c_{i3}(c_{11}'L_i + c_{12}'M_i + c_{13}'N_i), && (3)\\
Dm_1 &= \Sigma c_{i1}(c_{21}'L_i + c_{22}'M_i + c_{23}'N_i), && (4)\\
Dm_2 &= \Sigma c_{i2}(c_{21}'L_i + c_{22}'M_i + c_{23}'N_i), && (5)\\
Dm_3 &= \Sigma c_{i3}(c_{21}'L_i + c_{22}'M_i + c_{23}'N_i), && (6)\\
D\, n_1 &= \Sigma c_{i1}(c_{31}'L_i + c_{32}'M_i + c_{33}'N_i), && (7)\\
D\, n_2 &= \Sigma c_{i2}(c_{31}'L_i + c_{32}'M_i + c_{33}'N_i), && (8)\\
D\, n_3 &= \Sigma c_{i3}(c_{31}'L_i + c_{32}'M_i + c_{33}'N_i). && (9)
\end{aligned}
$$

Multiply the equations (1), (2), (3) by C_{11}, C_{12}, C_{13}, respectively, and add; then by C_{21}, C_{22}, C_{23}, and add; then by C_{31}, C_{32}, C_{33}, and add; the three results, after taking out the factor $C^{\frac{1}{2}}$, are

$$
\begin{aligned}
C'^{\frac{1}{2}}(C_{11}l_1 + C_{12}l_2 + C_{13}l_3) &= C^{\frac{1}{2}}(c_{11}'L_1 + c_{12}'M_1 + c_{13}'N_1), && (10)\\
C'^{\frac{1}{2}}(C_{21}l_1 + C_{22}l_2 + C_{23}l_3) &= C^{\frac{1}{2}}(c_{11}'L_2 + c_{12}'M_2 + c_{13}'N_2), && (11)\\
C'^{\frac{1}{2}}(C_{31}l_1 + C_{32}l_2 + C_{33}l_3) &= C^{\frac{1}{2}}(c_{11}'L_3 + c_{12}'M_3 + c_{13}'N_3). && (12)
\end{aligned}
$$

[1] Cf. Staude, An. Geom. des Punktes, etc., p. 159, p. 174.

The same multipliers applied in the same way to the equations (4), (5), (6); and finally to the equations (7), (8), (9), give

$$
\begin{aligned}
C'^{\frac{1}{2}}(C_{11}m_1 + C_{12}m_2 + C_{13}m_3) &= C^{\frac{1}{2}}(c_{21}'L_1 + c_{22}'M_1 + c_{23}'N_1), &(13)\\
C'^{\frac{1}{2}}(C_{21}m_1 + C_{22}m_2 + C_{23}m_3) &= C^{\frac{1}{2}}(c_{21}'L_2 + c_{22}'M_2 + c_{23}'N_2), &(14)\\
C'^{\frac{1}{2}}(C_{31}m_1 + C_{32}m_2 + C_{33}m_3) &= C^{\frac{1}{2}}(c_{21}'L_3 + c_{22}'M_3 + c_{23}'N_3), &(15)\\
C'^{\frac{1}{2}}(C_{11}n_1 + C_{12}n_2 + C_{13}n_3) &= C^{\frac{1}{2}}(c_{31}'L_1 + c_{32}'M_1 + c_{33}'N_1), &(16)\\
C'^{\frac{1}{2}}(C_{21}n_1 + C_{22}n_2 + C_{23}n_3) &= C^{\frac{1}{2}}(c_{31}'L_2 + c_{32}'M_2 + c_{33}'N_2), &(17)\\
C'^{\frac{1}{2}}(C_{31}n_1 + C_{32}n_2 + C_{33}n_3) &= C^{\frac{1}{2}}(c_{31}'L_3 + c_{32}'M_3 + c_{33}'N_3). &(18)
\end{aligned}
$$

Again, multiply the equations (1), (4), (7) by C_{11}', C_{21}', C_{31}', respectively, and add; then by C_{12}', C_{22}', C_{32}', and add; then by C_{13}', C_{23}', C_{33}', and add; the three results, after taking out the factor $C'^{\frac{1}{2}}$, are

$$
\begin{aligned}
C^{\frac{1}{2}}(C_{11}'l_1 + C_{21}'m_1 + C_{31}'n_1) &= C'^{\frac{1}{2}}(c_{11}L_1 + c_{21}L_2 + c_{31}L_3), &(19)\\
C^{\frac{1}{2}}(C_{12}'l_1 + C_{22}'m_1 + C_{32}'n_1) &= C'^{\frac{1}{2}}(c_{11}M_1 + c_{21}M_2 + c_{31}M_3), &(20)\\
C^{\frac{1}{2}}(C_{13}'l_1 + C_{23}'m_1 + C_{33}'n_1) &= C'^{\frac{1}{2}}(c_{11}N_1 + c_{21}N_2 + c_{31}N_3). &(21)
\end{aligned}
$$

The same multipliers applied to the equations (2), (5), (8), and then to the equations (3), (6), (9) give

$$
\begin{aligned}
C^{\frac{1}{2}}(C_{11}'l_2 + C_{21}'m_2 + C_{31}'n_2) &= C'^{\frac{1}{2}}(c_{12}L_1 + c_{22}L_2 + c_{32}L_3), &(22)\\
C^{\frac{1}{2}}(C_{12}'l_2 + C_{22}'m_2 + C_{32}'n_2) &= C'^{\frac{1}{2}}(c_{12}M_1 + c_{22}M_2 + c_{32}M_3), &(23)\\
C^{\frac{1}{2}}(C_{13}'l_2 + C_{23}'m_2 + C_{33}'n_2) &= C'^{\frac{1}{2}}(c_{12}N_1 + c_{22}N_2 + c_{32}N_3), &(24)\\
C^{\frac{1}{2}}(C_{11}'l_3 + C_{21}'m_3 + C_{31}'n_3) &= C'^{\frac{1}{2}}(c_{13}L_1 + c_{23}L_2 + c_{33}L_3), &(25)\\
C^{\frac{1}{2}}(C_{12}'l_3 + C_{22}'m_3 + C_{32}'n_3) &= C'^{\frac{1}{2}}(c_{13}M_1 + c_{23}M_2 + c_{33}M_3), &(26)\\
C^{\frac{1}{2}}(C_{13}'l_3 + C_{23}'m_3 + C_{33}'n_3) &= C'^{\frac{1}{2}}(c_{13}N_1 + c_{23}N_2 + c_{33}N_3). &(27)
\end{aligned}
$$

Multiply the equations (19), (22), (25) by C_{11}, C_{12}, C_{13}, respectively, and add; then by C_{21}, C_{22}, C_{23}, and add; then by C_{31}, C_{32}, C_{33}, and add; apply the same multipliers in the same way to (20), (23), (26); and then to (21), (24), (27); the results, after taking out the factor $C^{\frac{1}{2}}$ and inverting the sides of the equations, are ($i = 1, 2, 3$)

$$
\begin{aligned}
D\,L_1 &= \Sigma C_{1i}(C_{11}'l_i + C_{21}'m_i + C_{31}'n_i), &(28)\\
D\,L_2 &= \Sigma C_{2i}(C_{11}'l_i + C_{21}'m_i + C_{31}'n_i), &(29)\\
D\,L_3 &= \Sigma C_{3i}(C_{11}'l_i + C_{21}'m_i + C_{31}'n_i), &(30)\\
DM_1 &= \Sigma C_{1i}(C_{12}'l_i + C_{22}'m_i + C_{32}'n_i), &(31)\\
DM_2 &= \Sigma C_{2i}(C_{12}'l_i + C_{22}'m_i + C_{32}'n_i), &(32)\\
DM_3 &= \Sigma C_{3i}(C_{12}'l_i + C_{22}'m_i + C_{32}'n_i), &(33)\\
DN_1 &= \Sigma C_{1i}(C_{13}'l_i + C_{23}'m_i + C_{33}'n_i), &(34)\\
DN_2 &= \Sigma C_{2i}(C_{13}'l_i + C_{23}'m_i + C_{33}'n_i), &(35)\\
DN_3 &= \Sigma C_{3i}(C_{13}'l_i + C_{23}'m_i + C_{33}'n_i). &(36)
\end{aligned}
$$

The nine equations (1) to (9) give each element of D linearly in terms of the cofactors; the nine equations (28) to (36) give each cofactor linearly in terms of the elements themselves; the set of equations (10) to (18), and also the set (19) to (27), are linear expressions in the elements and cofactors of a row or column of D. The symmetry of the set (28)–(36) to the set (1)–(9) is noteworthy. Also that of the set (19)–(27) inverted to the set (10)–(18).

Any one of the four sets of nine equations (1)–(9), or (10)–(18), or (19)–(27), or (28)–(36), corresponds to the nine so-called Français[1] formulas for orthogonal systems, namely: Each element in D multiplied by D equals its cofactor.

[1] *Journ. Ecol. Pol.*, 1808.

As a check upon the work, the identities (10)–(27) can also be obtained by direct multiplication. For example, represent

$$\begin{vmatrix} \lambda_1' & \mu_1' & \nu_1' \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix}$$

by B_1; then, in the identity $A' \cdot (A \cdot B_1) = A \cdot (A' \cdot B_1)$, first on the left hand side multiply together the determinants A and B_1, and on the right hand side A' and B_1; the result is one of the identities (10)–(27).

The identities (10)–(27) are often useful in changing the form of the equations for the transformation from one system of oblique axes to another oblique system; for example, the equations (2) on p. 197 of Grunert, *Arch. d. Math.*, 34; also in proving the relation between the formulas here given and the formulas, for example, of Grunert and Sturm.

QUESTIONS AND DISCUSSIONS.

EDITED BY U. G. MITCHELL, University of Kansas.

NEW QUESTIONS.

25. In an investigation in physics Mr. Mason E. Hufford, 525 S. Park Ave., Bloomington, Indiana, has need of the values of the Bessel functions $J_0(x)$ and $J_1(x)$ for positive real values of x up to $x = 100$. Have tables been constructed to this extent? What is the most ready means by which the desired values may be computed to any required degree of accuracy?

REPLIES.

24. The following facts are significant:

(1) The New England Association of Mathematics Teachers has appointed a committee "to investigate the current criticisms of high school mathematics."

(2) A committee of the Council of the American Mathematical Society has under consideration the question "whether any action is desirable on the part of the Society in the matter of the movement against mathematics in the schools."

(3) At the recent meeting in Cincinnati of the National Education Association an iconoclastic discussion on the topic: "Can algebra and geometry be reorganized so as to justify their retention for high school pupils not likely to enter technical schools?" aroused approbation and applause. An outline of the remarks by one of the speakers is printed below.

In view of these facts what should be done by those who believe in the value of mathematics as a general high school study?

MEMORANDUM OF REMARKS MADE BY COMMISSIONER DAVID SNEDDEN, of Massachusetts, before the Mathematics Section of the Commission on the Reorganization of Secondary School Studies, Cincinnati, Ohio, February 25th, 1915.

Commissioner Snedden followed in his discussion the presentation of a paper from Assistant Superintendent Wicher, of New Hampshire, on the subject: "Can Algebra and Geometry be Reorganized so as to justify their Retention for Pupils Not Likely to Enter Technical Schools?" He said in substance:

(*a*) That at present algebra and geometry occupy substantially monopolistic positions in the curricula of secondary schools, in that in many such schools they

are required for graduation, and further, in that they are usually required for admission to college;

(*b*) That at the present time there is much uncertainty as to the actual educational values of these studies, more especially for girls, and that the burden of proof rests upon those who desire to have them further continued as prescriptive studies;

(*c*) That the great majority of teachers of mathematical subjects in high schools are not informed as to the ultimate educational values of these studies. They content themselves with pursuing proximate aims, among which are the mastery of the matter contained in the text-book, enabling students to meet closing examinations or college entrance examinations, etc.

(*d*) That among writers and thinkers on the subject of instruction in mathematics there seem to be five prominent ultimate aims held forth as justifying the importance attached to these studies in secondary schools, namely:

1. The so-called "disciplinary aim."
2. The so-called "instrumental aim," for students going into engineering and other callings requiring advanced mathematical knowledge.
3. The other so-called "instrumental aim," for students pursuing advanced studies in such fields as physics, economics, etc.
4. The first so-called "cultural aim," in which it is held that mathematical studies serve to interpret for the student the world in which he lives.
5. The second so-called "cultural aim," to the effect that the study of mathematics enables the student to interpret a substantial part of his social inheritance.

(*e*) That of the foregoing aims, that which has the largest support is the so-called "disciplinary aim." The value of mathematical studies under this head, however, has been seriously called into question by recent investigations, and the majority of the students of education are now inclined to attach relatively little value to it, holding that mental discipline must be a by-product of any and all studies pursued because of their ultimate worth in fitting the student for life in its cultural, civic and vocational aspects.

(*f*) That second in importance as the aim of mathematics, in the opinion of writers and speakers, is the claim that it serves in an instrumental capacity both for vocations and for subsequent higher studies. It is believed that the importance of this aim has been greatly exaggerated, and that the whole subject deserves special study, particularly as to the needs of girls.

(*g*) That each of the so-called "cultural" aims is important, but it is questionable whether they are in any substantial degree realized by present methods of teaching, it being contended that present methods of teaching, including thereunder the scope and character of material used, as found in text-books, etc., have been dictated by the requirements of the instrumental and disciplinary aims, and that as a consequence the large majority of high school students, as the outcome of their study of algebra and geometry, have gained neither in comprehension of the world in which they live nor in substantial appreciation of the mathematical portion of the so-called "social inheritance."

(h) That the most needed reform in the teaching of mathematics in secondary schools at the present time is an analytical consideration of the subject from the standpoint, first, of its instrumental or vocational use, and, second, its importance as an agency of cultural education. The high school (and equally the college) should then offer as electives quite technical courses in mathematics as an instrumental study, to those desiring the same. These schools should also offer, probably as electives, distinctly cultural courses designed to acquaint young people with the evolution of mathematical studies and the part they now play in the life of the individual and of society, these studies to be much more descriptive and interpretive than anything now found under the heads of algebra and geometry, even including the more historical treatment of these subjects.

DISCUSSIONS.

Relating to the use of i and $e^{\phi i}$ in vector-notation.

(Cf. article "A Note on Plane Kinematics," by Alexander Ziwet and Peter Field, this MONTHLY, Vol. XXI, pp. 105–113.)

DISCUSSION BY EDWIN BIDWELL WILSON, Massachusetts Institute of Technology.

In the MONTHLY, April, 1914, Ziwet and Field gave a very interesting and exceedingly neat treatment of certain elementary properties of plane kinematics based on the notation of Burali-Forti and Marcolongo, and they interpreted the analysis in a series of fundamental constructions. The treatment is elegant not alone by what it does, but particularly by what it leaves unsaid. The fundamental operation is multiplication by i or by $e^{\phi i}$, to turn a vector through the angle $\frac{1}{2}\pi$ or φ respectively. It is not mentioned, except once parenthetically, that, used in this sense, i and $e^{\phi i}$ are very different from the ordinary $i = \sqrt{-1}$ and $e^{\phi i} = \cos\varphi + \sqrt{-1}\sin\varphi$ of algebra. Indeed it is only when these operators are applied successively to the same vector that the analogy with complex numbers is valid. When applied in products of vectors the operators are not at all scalar. This may be seen in the following parallel columns, where I use the notations $\times$ and $\wedge$ for scalar and vector products (instead of the respective $\cdot$ and $\times$ of Gibbs).

i and $e^{\phi i}$ as operators	$i = \sqrt{-1},\quad e^{\phi i} = \cos\varphi + \sqrt{-1}\sin\varphi$
$i\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{a} \times i\boldsymbol{a} = 0$	$i\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{a} \times i\boldsymbol{a} = i\boldsymbol{a}^2$
$i\boldsymbol{a} \wedge \boldsymbol{a} = -\boldsymbol{a} \wedge i\boldsymbol{a} \neq 0$	$i\boldsymbol{a} \wedge \boldsymbol{a} = \boldsymbol{a} \wedge i\boldsymbol{a} = 0$
$i\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{a} \times i\boldsymbol{b}$	$i\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \times i\boldsymbol{b}$
$i\boldsymbol{a} \times i\boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}$	$i\boldsymbol{a} \times i\boldsymbol{b} = -\boldsymbol{a} \times b$
$e^{i\phi}\boldsymbol{a} \times e^{i\phi}\boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}$	$e^{i\phi}\boldsymbol{a} \times e^{i\phi}\boldsymbol{b} = e^{2i\phi}\boldsymbol{a} \times \boldsymbol{b}$
$\boldsymbol{a} \times e^{i\phi}\boldsymbol{b} = \boldsymbol{b} \times e^{-i\phi}\boldsymbol{a}$	$\boldsymbol{a} \times e^{i\phi}\boldsymbol{b} = \boldsymbol{b} \times e^{i\phi}\boldsymbol{a}$
$e^{i\phi}\boldsymbol{a} \times e^{i\psi}\boldsymbol{a} = \boldsymbol{a}^2\cos(\psi - \varphi)$	$e^{i\phi}\boldsymbol{a} \times e^{i\psi}\boldsymbol{a} = e^{i(\phi+\psi)}\boldsymbol{a}^2$
$e^{i\phi}\boldsymbol{a} \wedge e^{i\psi}\boldsymbol{a} = \boldsymbol{a}^2\sin(\psi - \varphi)\boldsymbol{u}$	$e^{i\phi}\boldsymbol{a} \wedge e^{i\psi}\boldsymbol{a} = 0.$

In the last row $\boldsymbol{u}$ is a unit vector perpendicular to the plane we are considering.

Now it seems for me that the simple directness of the Ziwet-Field analysis depends largely on the fact that the authors do not investigate the properties of their operator but operate (with one exception) with the assumed rules and knowledge of ordinary scalar complex numbers—a process which does not lead them into error because they do not need to go far enough to get into radical differences with ordinary algebra. It is, however, quite easy to accomplish all their work without introducing any new operators and without laying oneself open to error by using algebraic notations in non-algebraic senses. If $\boldsymbol{u}$ is a unit vector normal to the plane, an operator which turns a vector through a right angle is $\boldsymbol{u}\wedge$. Thus $\boldsymbol{u}\wedge\boldsymbol{a}$ is equal to $\boldsymbol{a}$ in magnitude and perpendicular to $\boldsymbol{a}$ in direction. The operator applied successively to the same vector gives by geometry the ordinary rule for powers of $\sqrt{-1}$; namely,

$$\boldsymbol{u}\wedge(\boldsymbol{u}\wedge\boldsymbol{a}) = -\boldsymbol{a}, \qquad \boldsymbol{u}\wedge(\boldsymbol{u}\wedge(\boldsymbol{u}\wedge\boldsymbol{a})) = -\boldsymbol{u}\wedge\boldsymbol{a}, \quad \cdots.$$

This also follows algebraically by the rule for the reduction of double, triple, $\cdots$, vector products.

We may go further and consider the operator

$$\begin{aligned} e^{\phi\boldsymbol{u}\wedge} &= 1 + \varphi\boldsymbol{u}\wedge + \frac{\varphi^2\boldsymbol{u}\wedge(\boldsymbol{u}\wedge}{2!} + \frac{\varphi^3\boldsymbol{u}\wedge(\boldsymbol{u}\wedge(\boldsymbol{u}\wedge}{3!} + \cdots \\ &= \left(1 - \frac{\varphi^2}{2!} + \cdots\right) + \left(\varphi - \frac{\varphi^3}{3!} + \cdots\right)\boldsymbol{u}\wedge \\ &= \cos\varphi + \sin\varphi\,\boldsymbol{u}\wedge. \end{aligned}$$

We should thus see that we get a formula entirely analogous to that for $e^{\phi i}$; but we should not be tempted to think of the operator as a scalar operator. We should realize at each step that we were dealing with a vector operator. We should, moreover, be right in line with the more general work of A. C. Lunn who, in the Monthly, February, 1909, has given a very elegant derivation of the Euler-Rodrigues form for a rotation based upon the operator $\boldsymbol{u}\wedge$ applied without the restriction that the operand should be perpendicular to $\boldsymbol{u}$. Indeed the equations

$$\boldsymbol{u}\wedge(\boldsymbol{u}\wedge\boldsymbol{r}) = (\boldsymbol{u}\times\boldsymbol{r})\boldsymbol{u} - \boldsymbol{r}, \qquad \boldsymbol{u}\wedge(\boldsymbol{u}\wedge(\boldsymbol{u}\wedge\boldsymbol{r})) = -\boldsymbol{u}\wedge\boldsymbol{r}, \quad \cdots$$

show that the general form of the operator is

$$e^{\phi\boldsymbol{u}\wedge} = \cos\varphi + \sin\varphi\boldsymbol{u}\wedge + (1-\cos\varphi)\boldsymbol{u}(\boldsymbol{u}\times.$$

Although I admit the neatness of the i and $e^{\phi i}$ introduced by Burali-Forti and Marcolongo and adopted by Ziwet and Field, I do feel that the elegance thus obtained does not more than make up for the dangers accompanying the notations, to say nothing of the restriction of the operand to a plane and the suppression from the analysis of the vital vector $\boldsymbol{u}$. Is it not likely that a large amount of the appeal of i and $e^{\phi i}$ in this sense is due to our neglect of the operational calculus?

Those who have a little knowledge of this calculus, as applied in solving linear differential equations with constant coefficients, in determining the Bernoulli numbers, and the like,[1] should welcome a chance to reveal, rather than to obscure, that method in the present problem. And to anyone who writes Taylor's expansion for $f(x)$ as $e^{hD}f(x)$, the significance of $e^{\phi u \wedge}$ is apparent.

There is another objection to the specialized non-algebraic use of i and $e^{\phi i}$ as operators and this lies in the fact that their ordinary algebraic use is of importance in the theory of complex vectors.[2] A complex vector or bivector, as Gibbs called it, is written $\boldsymbol{A} + i\boldsymbol{B}$, where $i = \sqrt{-1}$, and is represented by the two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ taken together. The expression $e^{\phi i}(\boldsymbol{A} + i\boldsymbol{B})$ treated algebraically gives

$$e^{\phi i}(\boldsymbol{A} + i\boldsymbol{B}) = (\boldsymbol{A} \cos \varphi - \boldsymbol{B} \sin \varphi) + i(\boldsymbol{A} \sin \varphi + \boldsymbol{B} \cos \varphi).$$

If we consider $\boldsymbol{A}$ and $\boldsymbol{B}$ drawn from the same origin, the vectors

$$\boldsymbol{A}' = \boldsymbol{A} \cos \varphi - \boldsymbol{B} \sin \varphi, \qquad \boldsymbol{B}' = \boldsymbol{A} \sin \varphi + \boldsymbol{B} \cos \varphi,$$

drawn from the same origin, terminate on the ellipse constructed with $\boldsymbol{A}$ and $\boldsymbol{B}$ as conjugate radii, and in particular, if we consider rotation from $\boldsymbol{A}$ to $\boldsymbol{B}$ positive, the vectors $\boldsymbol{A}'$ and $\boldsymbol{B}'$ are set back relative to $\boldsymbol{A}$ and $\boldsymbol{B}$ by the (eccentric) angle φ. Thus multiplication by $e^{\phi i}$ has a simple geometric meaning.

Particularly simple are the circular bivectors, namely, bivectors in which $\boldsymbol{A}$ and $\boldsymbol{B}$ are perpendicular and of equal magnitude. Multiplication by $e^{\phi i}$ is in this case ordinary rotation through the angle $-\varphi$. Following the method used by Gibbs in optics we may represent relative motion in a plane by a circular bivector formed by adding to the position vector $\boldsymbol{p} = P - 0$, the velocity $\dot{\boldsymbol{p}}$ multiplied by $i = \sqrt{-1}$ and divided by the angular velocity w. We have

$$\boldsymbol{M} = \boldsymbol{p} + \frac{i}{w}\dot{\boldsymbol{p}}.$$

The motion of all other points in the plane may then be expressed as

$$\boldsymbol{M}' = re^{-\phi i}\boldsymbol{M},$$

where r and φ are polar coördinates of any point with O as pole, $\boldsymbol{p}$ as polar axis and mod $\boldsymbol{p}$ as unit of distance. We shall not try to reproduce the work of Ziwet and Field, but shall content ourselves with having stated that the ordinary $i = \sqrt{-1}$ is interesting and useful in vector analysis, especially in applications to optics.

Note.—It may be worth while, in connection with this Discussion, to add a reference to A. Macfarlane's paper on "A System of Notation for Vector-Analysis; with a Discussion of the Underlying Principles," published by The International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics (Press of The New Era Printing Co., June,

[1] See Wilson's *Advanced Calculus*, pp. 149–152, 214–224, 275, 447–449, and passages in Heaviside's *Electromagnetic Theory*.

[2] Gibbs-Wilson, *Vector Analysis*, pp. 426ff.

1912), and to recall that the first two of the fifteen conditions mentioned by him as desirable in a systematic and adequate vector-notation were (in part):

1. It should disturb as little as possible the established notation of mathematical analysis. For example, i should be left to denote $\sqrt{-1}\cdots$.
2. It may generalize, but not contradict, the notation and principles of algebra.—EDITOR.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY, University of Minnesota.

TWO NEW TRIGONOMETRIES.

Plane Trigonometry and Tables. By GEORGE WENTWORTH and DAVID EUGENE SMITH. Ginn and Company, Boston, 1914. iv+188 and v+104 pages.

This text is a revised version of the old and widely known book of similar content of the "Wentworth series." With a few exceptions it consists of a partial rearrangement of a portion of the material of the older book, and users of the older book will have no difficulty in recognizing the text and many of the problems of the newer book. Besides some new problems, there have been added a few notes of a historical character, a chapter on logarithms and a chapter on graphs. In this latter chapter the graphs of some of the simpler algebraic functions as well as those of the six trigonometric functions are given. Why does not some author accidentally introduce a new idea into trigonometry by giving the figure for $y = \sin 2x$ or $y = \sin x + \cos x$? The broad treatment of trigonometric equations and identities, and the introduction of the radian are deferred to the end of the text. In its new dress the text will probably lose no old friends and may gain some new ones.

Plane Trigonometry with Tables. By C. I. PALMER and C. W. LEIGH. The McGraw-Hill Book Company, New York, 1914. x+156+132 pages.

The authors make no claim for novelty, new material, or any innovation in this book. They hope to give a better, clearer arrangement of the time-worn subject matter. It is noticeable that the explanation of logarithms forms a separate chapter, bound with the tables, and that in the trigonometry proper there is no crowding of their use into any particular place. The type chosen for the tables seems unfortunate, being tiresome to the eyes if the tables are used for any length of time.

The introductory chapter contains adequate and particularly clear explanations of directed angles and lines together with the graphical addition of each. With this as a basis and with the use of the ideas of rectangular and polar coördinates, the trigonometric functions of the general angle are defined and the simpler developments presented. After this has been done successfully, the special definitions of the trigonometric functions for the right-angled triangle are dragged in from nowhere in particular and for no particularly good reason, and then they are used a great deal. The simpler identities and the inverse functions, involving acute angles, are then introduced.

The following chapter, devoted to the right triangle, introduces the ideas of orthogonal projection, vectors and areas of sectors and segments. The authors assume that the angles used in these latter discussions are less than a right angle, which is of course very unfortunate. By using the graphs of the trigonometric functions, the authors lead up to the idea of the addition theorems, and then in a natural way introduce the idea of the many-valuedness of an inverse function. In most places from this point on, only the principal value of the inverse function is used.

In the discussion of the general addition theorems, the proofs are given for *admittedly* special cases and no attempt is made to mislead the student into thinking that the proofs given are general. Some indications are given for the extension of the proofs. Identities, direct and inverse, based on these theorems conclude the chapter. Then come chapters on oblique triangles, trigonometric equations, and De Moivre's theorem including hyperbolic functions. In this latter chapter only the elements of the hyperbolic functions are given, the idea apparently being to give some place to which to refer when the future need for these functions shall occur. In the discussion of the De Moivre theorem, the authors use $j = \sqrt{-1}$ which is extremely unfortunate even though the book is primarily for technical students and $a + jb$ is the standard representation of a complex number in electrical theory.

The number of problems, many of which come from the applications, is large. The arrangement of the material in the chapters is generally good and natural. A rearrangement of the chapters themselves, so as to defer the solution of all triangles until the end of the book, seems desirable, the idea being that *occasionally* trigonometry is taught by teachers who have no proper perspective as to the future need for trigonometry in mathematics, over-emphasis being frequently given to the solution of triangles and more important subjects skimped.

C. F. Craig

Geometrical Researches on the Theory of Parallels. By Nicholaus Lobatschevski. Translated from the original by George Bruce Halsted. New Edition. The Open Court Publishing Company, Chicago, 1914. 50 pages. $1.25.

This is a new edition of a work which appeared in 1891. The desirability that such classics of mathematical literature should be available even to those students who do not have command of any foreign language is recognized by mathematicians the world over. France and Germany have been unusually progressive in such matters, and Italy is just inaugurating a series of the classics. The beautiful simplicity of these masterpieces is a source of encouragement to young students to continue their studies. When a real interest is aroused in non-euclidean geometry or other topic, the study soon extends to the various fields of mathematical research which are so closely and so curiously inter-related. The student of the history of science is particularly interested in the fact that the names of Bolyai and Lobatschevski, independent workers, are indissolubly linked

with this wonderful development of geometry, while Gauss and even older writers like Saccheri were on the brink of the discovery. The possibility of the simultaneous discovery was not due entirely to the genius, great though it was, of the two fortunate individuals mentioned, but also to the many humble and obscure scientists who paved the way.

For those who desire to pursue further the subject of the book under discussion, H. S. Carslaw's translation of Roberto Bonola's "Non-Euclidean Geometry" is to be commended. It is published by the Open Court Publishing Company. Manning's "Non-Euclidean Geometry," published by Ginn and Company, Coolidge's "Elements of Non-Euclidean Geometry," published at Oxford in 1909, and Somerville's work with the same title are other available English books on the subject.

The paper and the printing of the book seem to be of distinctly inferior quality compared with that usually found in books with the same imprint. Some of the work seems to be from the old plates. The price is excessive for a fifty-page book. " Bibliography " is used in a misleading sense, as there is no bibliography given. The quotation in French on page 9 needs revision or explanation.

L. C. KARPINSKI.

Our Knowledge of the External World as a Field for Scientific Method in Philosophy. By BERTRAND RUSSELL. ix+245 pages. The Open Court Publishing Co., Chicago, 1914. $2.00.

The author distinguishes three types among present-day philosophies. The first of these, which he calls the classical tradition, descends in the main from Kant and Hegel. The second type, which is called evolutionism, derives its predominance from Darwin and reckons Herbert Spencer as its first philosophical representative but in recent times has been largely modified by William James and Henri Bergson. " The third type, which may be called logical atomism for want of a better name," says our author, " has gradually crept into philosophy through the critical scrutiny of mathematics. This type of philosophy . . . represents, I believe, the same kind of advance as was introduced into physics by Galileo: the substitution of piecemeal, detailed, and verifiable results for large untested generalities recommended only by a certain appeal to imagination."

It is to the third type of philosophy that the volume under review belongs. It should be of considerable interest to mathematicians not only from the fact that the methods employed have arisen from a critical scrutiny of mathematics but also because much of its detailed treatment is obviously inspired by the mathematical ideas and results of the past forty years.

This is not the place for a detailed review of the book, even though it is an important one. It should have a large circle of readers among both mathematicians and philosophers. On the work of the latter it will probably exert a wide influence.

R. D. CARMICHAEL.

Problems of Science. By FEDERIGO ENRIQUES. Authorized translation by KATHARINE ROYCE with an introductory note by JOSIAH ROYCE. xvi+392 pages. The Open Court Publishing Co., Chicago, 1914. $2.50.

The first edition of the Italian text of the *Problema della Scienza* of Professor Enriques appeared in 1906. It has already become known to a wide circle of European students. It is a pleasure to welcome its appearance in English.

The book contains six chapters treating in order the following topics: the general problem of knowledge and related matters; facts and theories and their interactions; the general problems of logic; the philosophical and psychological questions which are naturally raised in connection with the science of geometry; mechanics, its objective significance and the psychological development of its principles; the extension of mechanics into physics and the relation of the mechanical hypothesis to the phenomena of life.

This book is of quite unusual value. It is written by a mathematician and consequently takes proper account of recent mathematical developments. It contains a masterly analysis of the problems of science, especially in their relation to matters of mathematical expression and of philosophical import. It merits the close attention alike of physicists and philosophers and mathematicians.

R. D. CARMICHAEL.

Geometry of Four Dimensions. By HENRY PARKER MANNING, Ph.D., Associate Professor of Pure Mathematics in Brown University. The Macmillan Company, New York, 1914. ix+348 pages. $2.00.

The production of this work, which has taken several years, has evidently been a labor of love. The result is a book physically handsome, beautiful in its content, creditable alike to the publishers and to the author. The introduction of 22 pages is itself worth more than the price of the volume. We have here, besides a clear indication of the aim and point of view of the present work along with reasons for its methods and procedure, a fair evaluation of hyperspace studies in general, and an admirable sketch of the origin and earlier developments of the subject with a clew to its literature to date. It is a curious fact that, although space dimensionality was a subject of thought with philosophers and mathematicians from Aristotle down, yet geometry of hyperspace is still under a hundred years old, the earliest contribution to the subject being, Professor Manning tells us, that of Möbius in his *Calcul*, 1827, and even Möbius thought that 4-dimensional space could not be "*gedacht*." Why was the beginning so tardy? "The general notion," says Professor Manning, "that geometry is concerned only with objective external space made the existence of any kind of geometry seem to depend upon the existence of the same kind of space." The answer is good so far as it goes but it might have gone farther and deeper. It might well have been made clear that there neither is nor can be a mathematical geometry of sensible space; that the subject of mathematical geometry is indeed objective external space but is a conceptual space of the kind and not a sensible one; and that

the space of 4-dimensional or n-dimensional geometry has the same kind of existence as has the space of ordinary solid geometry. What prevented the rise of multi-dimensional geometry was, not bad logic, but bad psychology; and it is bad psychology that still leads many mathematicians to apologize for using the language of such geometry because it seems to imply belief in the existence of a corresponding space. The philosophy of mathematics has had many devotees. Of the logic of mathematics there is a large and growing literature. The psychology of mathematics has been neglected. Here is a great opportunity. Who will improve it?

"Our plane and solid geometries," says Professor Manning, "are but the beginnings of this science" (of geometry). He is right. Just as a student learns the real meaning of analysis only when he studies the calculus, so the real nature of geometry shows itself only in the light of the theory of hyperspace and especially that of four dimensions. The book in hand provides a natural transition. The methods are those of the ordinary geometry of the secondary school or of college, and no mathematical knowledge is presupposed beyond that of the usual elements of solid geometry. Algebra is not employed. The method is that of so-called pure or synthetic geometry. Beyond the explicit introduction of two axioms concerned with the relation of collinearity, no special stress is given to the matter of postulates or of rigor for rigor's sake. Instead, the student is presented with a four-dimensional structure built upon familiar foundations by familiar methods. In this respect and in its limitation to space of four dimensions, the work compares both favorably, as being simpler, and unfavorably, as being less rich, with the *Mehrdimensionale Geometrie* of Schoute, where hyperspaces of every dimensionality are studied by several elementary methods both analytic and synthetic. One who had read Schoute's book would not need to read that of Manning. On the other hand many a student not qualified to read the former can read the latter with pleasure and profit. The work is a point geometry in the sense that the point is the undefined element, all other figures being regarded as classes of points. The procedure being that of metric geometry, one misses the beautiful interplay of such dual developments as arise from the projective point of view out of the reciprocity of point and hyperlane, and of line and plane. The first 220 pages are pangeometric in the Lobachevskian sense, no question of parallels intervening in course of the first five chapters. Otherwise the theory is Euclidean except for some sections devoted to such themes as the geometry of the hypersphere and that of the hyperplane at infinity. It is notable, by the way, that the phrase "at infinity" and equivalent phrases are used merely to facilitate talk about parallelism and do not imply the existence of infinite numbers or distances, no infinite region or element being posited.

In the first chapter, concerned with foundations, certain familiar concepts and theorems are presented with extraordinary care and clarity to fortify the reader against the approaching shock of such 4-space relations and possibilities as transcend experience. Perhaps the author is a bit unfortunate in speaking of such relations—the fact, *e. g.*, that in general two planes have one and but one

common point—as "contradicting" experience. To say that it is transcendence rather than contradiction seems far enough from verbal quibbling. It seems a pity that the notion of a line pencil was not available so that the very beautiful theorem on page 61 could have been made to read: *Two independent planes contain, each of them, one and but one pencil of lines such that any two of the lines are coplanar.* Denote the planes by α and β. Then we may say with Professor Manning that "α and β are covered with these lines." But it is somewhat ambiguous to say that the planes "might be said to consist of them." They consist of the points of the lines but not of the lines as lines.

It is in the second chapter, devoted to perpendicularity, that the beginner will realize that the narrow shell of his familiar geometric conceptions is bursting asunder and letting in the light of vistas of which he had never dreamed. And the surprise and joy will attend him throughout the book, being indeed renewed and deepened when he enters upon the doctrine of parallelism in chapter VI.

The normal development is interrupted in chapter III in order to point out the possibility and nature of what is called "Point Geometry," that is, the theory of the angles at a 4-space point. This is a 3-dimensional geometry, not of, but *within*, 4-space. The name is hardly felicitous in view of the fact that the entity taken for element is not the point nor indeed the angle but the line or half-line. Possibly a better name would have been line geometry or angle geometry of a 4-space point. It might have been indicated, too, that there is possible a kind of reciprocal 3-dimensional geometry of the hyperplanes enveloping a point, as well as, matching the line geometry of a hyperplane, a 4-dimensional geometry of the planes enveloping a point of 4-space. Analogous suggestions occur in noticing the place here accorded to what the author calls "Edge Geometry," that is, the 2-dimensional theory of planes or half-planes having a line or edge in common.

Chapters IV, V, VI, VII, and VIII deal respectively with symmetry, order, and motion; hyperpyramids, hypercones, and the hypersphere; Euclidean geometry, figures with parallel elements; measurement of volumes and hypervolumes in hyperspace; and the regular polyhedroids.

The book ought to be in the library of every teacher of high-school mathematics and would serve admirably for use in an advanced undergraduate elective course in pure geometry.

CASSIUS J. KEYSER.

COLUMBIA UNIVERSITY.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

PROBLEMS FOR SOLUTION.

ALGEBRA.

When this issue was made up, no solution had been received for number 428.

430. Proposed by H. C. FEEMSTER, York College, Nebraska.

Solve the equations

$$\sum_{i=1}^{n} x_i - x_n = k + \frac{n^2 - 3n + 2}{2} d, \tag{1}$$

$$\sum_{i=1}^{n} x_i - x_{n-1} = k + \frac{n^2 - 3n + 4}{2} d, \tag{2}$$

$$\sum_{i=1}^{n} x_i - x_{n-2} = k + \frac{n^2 - 3n + 6}{2} d, \tag{3}$$

$$\cdots \quad \cdots \quad \cdots \quad \vdots$$

$$\sum_{i=1}^{n} x_i - x_1 = k + \frac{n^2 - n}{2} d. \tag{n}$$

431. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Form a magic square of 9 cells such that (the integers being all different) the products of the integers in the rows, columns, and diagonals shall be the same and the smallest product possible.

432. Proposed by C. N. SCHMALL, New York City.

There are n straight lines in a plane, no two of which are parallel, and no three of which are concurrent. Their points of intersection being joined, show that the number of new lines drawn is $\frac{1}{8}n(n-1)(n-2)(n-3)$.

GEOMETRY.

When this issue was made up, no solution had been received for number 459.

460. Proposed by J. W. CLAWSON, Ursinus College, Pa.

ABC is a triangle, O and I the centers of the circum- and in-circles respectively, and I', I'', I''' the centers of the three escribed circles. If AO, BO, CO meet the circumcircle in P, P', P'' respectively, and PR, $P'R'$, $P''R''$ are drawn parallel respectively to AI, BI, CI to meet BC, CA, AB respectively in R, R', R'', prove that: (1) PR, $P'R'$, $P''R''$ are concurrent, say at J. (2) $JO = OI$. (3) JI', JI'', JI''' are perpendicular respectively to BC, CA, AB. (4) AR, BR', CR'' are concurrent.

461. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Prove by means of *any* inscribed triangle the following trigonometrical relations: $\sin 2\beta = 2 \sin\beta \cos\beta$. $\cos 2\beta = \cos^2\beta - \sin^2\beta$. $\sin 3\beta = 3 \sin\beta - 4 \sin^3\beta$. $\cos 3\beta = 4 \cos^3\beta - 3 \cos\beta$.

462. Proposed by DANIEL KRETH, Wellman, Iowa.

A conical glass, the diameter of the base of which is 5 inches and altitude 6 inches, is one-fifth full of water. If a sphere 4 inches in diameter is dropped into it, how much of the vertical axis of the glass is immersed?

463. Proposed by NATHAN ALTSHILLER, University of Washington, Seattle.

Through a given point, to draw a line that cuts off on the sides of a given angle two segments the sum of which has a given value.

CALCULUS.

When this issue was made up, no solution had been received for number 378.

380. Proposed by C. N. SCHMALL, New York City.

Show that

$$\int_0^\infty \left[\frac{1}{1^4+x^2}+\frac{1}{2^4+x^2}+\frac{1}{3^4+x^2}+\cdots\right]dx=\frac{\pi^3}{12^3},$$

where the series in the brackets is infinite.

381. Proposed by ELBERT H. CLARKE, Purdue University.

Of all points having the same latitude and a constant difference α in their longitudes, to find the latitude of the two so situated that the distance between them, measured along their common parallel of latitude, shall exceed the distance between them measured on their great circle by the greatest possible amount.

382. Proposed by B. J. BROWN, Student in Drury College.

Discuss for what values of m and n, the integral, $\int_0^1 x^{m-1}(1-x)^{n-1}dx$, is finite and show how this integral can be expressed by means of integrals of the form $\int_0^\infty e^{-x}x^{p-1}dx$.

MECHANICS.

When this issue was made up, solutions had been received for numbers 297, 301, and 302.

304. Proposed by B. F. FINKEL, Drury College.

A spherical shell, inner radius r and outer radius R, has within it a perfectly smooth solid sphere of the same material and with radius $r_1 < r$. If the inner surface of the spherical shell is also perfectly smooth, determine the motion, after the time t, of the shell and sphere down a rough inclined plane, inclination α.

305. Proposed by B. J. BROWN, Student in Drury College.

A particle is to be projected so as to graze the top of a wall h feet high, at a distance of a feet from the point of projection, and to strike the ground at a distance b feet from the foot of the wall. Find the velocity of projection, and the inclination of the path to the horizontal, at the ground and at the top of the wall. I. C. S. 1903.

306. Proposed by EMMA M. GIBSON, Drury College.

A sphere is composed of a solid homogeneous hemisphere and a very thin hemispherical shell of equal mass.

What is the greatest inclination of a rough plane on which the sphere can just rest in equilibrium?

NUMBER THEORY.

When this issue was made up, no solutions had been received for numbers 227 and 228.

230. Proposed by E. B. ESCOTT, Ann Arbor, Michigan.

Find three numbers such that their sum, the sum of their squares, and the sum of their cubes, shall be a cube.

Note.—W. D. Cairns says this problem, which was proposed in *L'Intermediaire* in 1900, remains unsolved to date, even though it was reprinted in that journal in February, 1913.

231. Proposed by A. J. KEMPNER, University of Illinois.

Is the series whose terms are the reciprocals of all positive integers not containing a given combination of figures, for example not containing the combination 37, convergent or divergent?

Numbers such as $\frac{1}{37}, \frac{1}{370}, \frac{1}{5371}$ shall be omitted, numbers such as $\frac{1}{73}, \frac{1}{307}, \frac{1}{5317}$ shall be admitted as terms of the series. (Compare AMERICAN MATHEMATICAL MONTHLY, Volume XXI, page 123.)

SOLUTIONS OF PROBLEMS.

ALGEBRA.

421. Proposed by C. N. SCHMALL, New York City.

Give a trigonometrical solution of the general quadratic equation.

SOLUTION BY PAUL CAPRON, United States Naval Academy.

Let the equation be

$$ax^2 + bx + c = 0 \qquad (a > 0).$$

Then the algebraic solution

$$x = \frac{b}{2a}\left(-1 \pm \sqrt{1 - \frac{4ac}{b^2}}\right)$$

may be evaluated:

(1) If $c > 0$, let

$$\frac{4ac}{b^2} = \sin^2\theta;$$

then

$$x = \frac{b}{2a}(-1 \pm \cos\theta) = -\frac{b}{a}\sin^2\frac{\theta}{2} \qquad \text{or} \qquad -\frac{b}{a}\cos^2\frac{\theta}{2}.$$

(2) If $c < 0$, let

$$-\frac{4ac}{b^2} = \tan^2\phi;$$

then

$$x = \frac{b}{2a}(-1 \pm \sec\phi) = \frac{b}{a}\sec\phi\sin^2\frac{\phi}{2} \qquad \text{or} \qquad -\frac{b}{a}\sec\phi\cos^2\frac{\phi}{2}.$$

If tables of the hyperbolic functions are at hand, the following is simpler than (2):

(3) If $c < 0$, let

$$-\frac{4ac}{b^2} = \sinh^2 y;$$

then

$$x = \frac{b}{2a}(1 - \pm \cosh y) = \frac{b}{2a}\sinh^2 y/2 \qquad \text{or} \qquad -\frac{b}{2a}\cosh^2 y/2.$$

If a table of haversines is available, the following is simpler than (1):

(4) If $c > 0$, let

$$\frac{4ac}{b^2} = \sin^2\theta;$$

then

$$x = -\frac{b}{a}\operatorname{hav}\theta \qquad \text{or} \qquad -\frac{b}{a}\operatorname{hav}(180^\circ - \theta).$$

Also solved by the PROPOSER.

422. Proposed by W. D. CAIRNS, Oberlin College.

Find a solution of the equation $x^{x\sqrt{x}} = (x\sqrt{x})^x$. [Adapted from *Godfrey & Siddon's Elementary Algebra.*]

Solved by EDWARD S. INGHAM, F. L. CARMICHAEL, H. T. BIGLOW, ELIZABETH A. DAVIS. WALTER C. EELLS, D. H. RICHERT, H. C. FEEMSTER, A. H. HOLMES, FRANK IRWIN, GEORGE Y, SOSNOW, ELBERT H. CLARKE, A. M. HARDING, and the PROPOSER.

SOLUTION BY A. M. HARDING, University of Arkansas.

If we take the logarithm of both members of the equation, we obtain

$$x\sqrt{x}\log x = x\log(x\sqrt{x}) = x\log(x)^{\frac{3}{2}} = \tfrac{3}{2}x\log x,$$

or

$$x(\sqrt{x} - \tfrac{3}{2})\log x = 0.$$

Hence $x = 0$, $\log x = 0$, and $\sqrt{x} - \frac{3}{2} = 0$. Therefore $x = 0$, $x = 1$, and $x = \frac{9}{4}$. If we substitute $x = 0$, both members of the given equation take the indeterminate form 0^0. In order to show that $x = 0$ is a root, we proceed as follows:

Since $x^{x^{3/2}} = (x^{3/2})^x = x^{\frac{3}{2}x}$, we have $x^{(x^{3/2}-\frac{3}{2}x)} = 1$. Hence we must show that $[x^{(x^{3/2}-\frac{3}{2}x)}] \doteq 1$ as $x \doteq 0$. Let $y = x^{(x^{3/2}-\frac{3}{2}x)}$, then $\log y = (x^{3/2} - \frac{3}{2}x)\log x$, or $\log y = \dfrac{\log x}{(x^{3/2} - \frac{3}{2}x)^{-1}} \doteq \dfrac{\infty}{\infty}$ as $x \doteq 0$. Differentiate numerator and denominator. Then

$$\lim_{x\doteq 0}\frac{\log x}{(x^{3/2} - \frac{3}{2}x)^{-1}} = \lim_{x\doteq 0}\frac{-2(x^{3/2} - \frac{3}{2}x)^2}{3(x^{3/2} - x)} = \lim_{x\doteq 0}\frac{-2x(x^{1/2} - \frac{3}{2})}{3(x^{1/2} - 1)} = 0.$$

Hence $\lim\limits_{x\doteq 0} y = \lim\limits_{x\doteq 0}[x^{(x^{3/2}-\frac{3}{2}x)}] = 1$, and $x = 0$ is a root of the given equation.

423. Proposed by ELBERT H. CLARKE, Purdue University.

Show that the following formula is true for all positive integral values of k. The parenthetical symbols are defined as being the binomial coefficients, and $\binom{k+1}{0} = 1$, by definition.

$$k^k\binom{k+1}{0} - (k-1)^k\binom{k+1}{1} + (k-2)^k\binom{k+1}{2} + \cdots$$
$$+ (-1)^{s-1}(k-s+1)^k\binom{k+1}{s-1} + \cdots + (-1)^{k-1}\binom{k+1}{k-1} = 1.$$

I. SOLUTION BY A. M. HARDING, University of Arkansas.

There is a misprint in the last term of the left member of the equation as originally given. The exponent should be $k - 1$.

We have

$$e^{kx} = 1 + kx + \frac{k^2x^2}{2!} + \cdots + \frac{k^kx^k}{k!} + \cdots$$

$$e^{(k-1)x} = 1 + (k-1)x + \frac{(k-1)^2x^2}{2!} + \cdots + \frac{(k-1)^kx^k}{k!} + \cdots$$

$$\cdots = \cdots$$

$$e^{(k-s+1)x} = 1 + (k-s+1)x + \frac{(k-s+1)^2x^2}{2!} + \cdots + \frac{(k-s+1)^kx^k}{k!} + \cdots$$

$$\cdots = \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots.$$

Multiply these equations in order by $\binom{k+1}{0}, -\binom{k+1}{1}, \cdots,$

$$(-1)^{s-1}\binom{k+1}{s-1}, \cdots, (-1)^{k-1}\binom{k+1}{k-1} \qquad \text{and add.}$$

Denote the left member of the given equation by $F(k)$.

It will be seen that the coefficient of x^k in the above sum is $\frac{F(k)}{k!}$. Hence $\frac{F(k)}{k!}$ is the coefficient of x^k in the expansion

$$e^{kx} - \binom{k+1}{1}e^{(k-1)x} + \binom{k+1}{2}e^{(k-2)x} - \cdots$$

$$+ (-1)^{s-1}\binom{k+1}{s-1}e^{(k-s+1)x} + \cdots + (-1)^{k-1}\binom{k+1}{k-1}e^{x}.$$

This expansion can be written in the form

$$e^{kx}\left[1 - \binom{k+1}{1}e^{-x} + \binom{k+1}{2}e^{-2x} - \cdots + (-1)^{s-1}\binom{k+1}{s-1}e^{(-s+1)x}\right.$$

$$\left. + \cdots + (-1)^{k-1}\binom{k+1}{k-1}e^{-(k-1)x}\right]$$

$$= e^{kx}\left[(1-e^{-x})^{k+1} - (-1)^k\binom{k+1}{k}e^{-kx} - (-1)^{k+1}\binom{k+1}{k+1}e^{-(k+1)x}\right]$$

$$= e^{kx}(1-e^{-x})^{k+1} + (-1)^{k+1}(k+1-e^{-x})$$

$$= \left(1 + kx + \frac{k^2x^2}{2!} + \cdots\right)\left(x - \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots\right)^{k+1}$$

$$+ (-1)^{k+1}\left(k + x - \frac{x^2}{2!} + \cdots + (-1)^{k+1}\frac{x^k}{k!} + \cdots\right).$$

The coefficient of x^k in this expression is $\frac{(-1)^{2k+2}}{k!}$.

Hence $\frac{F(k)}{k!} = \frac{1}{k!}$. Hence $F(k) = 1$.

II. Solution by A. M. Kenyon, Purdue University.

This formula is a special case under the theorem:

If the coefficients of the binomial expansion of $(x-y)^n$, $n = 1, 2, 3, \cdots$, be multiplied, term by term, by the mth power, $m = 0, 1, 2, \cdots$, of the terms of any arithmetic progression, the sum of the products will vanish if $m < n$.

$$\sum_{t=0}^{n}(-1)^t\binom{n}{t}(a+td)^m = (-d)^m\sum_{t=0}^{n}(-1)^t\binom{n}{t}(x-t)^m,$$

where $x = -a/d$. The sum on the right is equal to

$$\sum_{s=0}^{m}(-1)^s\binom{m}{s}x^{m-s}\sum_{t=0}^{n}(-1)^t\binom{n}{t}t^s,$$

of which the inner sum vanishes for $s < n$, therefore for every s when $m < n$.

If now we set $k+1$ for n, k for a, -1 for d, and k for m, we have

$$\sum_{t=0}^{k+1}(-1)^t\binom{k+1}{t}(k-1)^k = 0,$$

that is,

$$\binom{k+1}{0}k^k - \binom{k+1}{1}(k-1)^k + \cdots + (-1)^{k-1}\binom{k+1}{k-1}1^k + 0 - 1 = 0.$$

Also solved by S. A. JOFFE.

GEOMETRY.

448. Proposed by S. W. REAVES, University of Oklahoma.

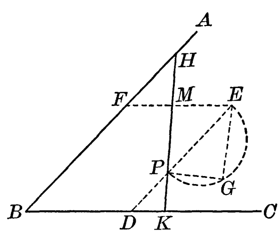
Through a given point P within a given angle to draw a line which shall form with the sides of the angle a triangle of a given area [Well's *New Plane Geometry* (1909), p. 153].

Solved in various ways by HORACE OLSON, CLIFFORD N. MILLS, GEORGE Y. SOSNOW, NATHAN ALTSHILLER, and A. HOLMES.

SOLUTION BY H. O. HANSON, East Elmhurst, N. Y.

Let $\angle ABC$ be the given angle, and P the given point within the angle.

Through P draw the line DE parallel to BA, and draw the line FE parallel to BC so that $BFED$ forms a parallelogram of the given area. On PE as hypothenuse construct the right triangle PEG so that $PG = PD$. Lay off $FH = GE$, and draw a line through H and P meeting BC at K and FE at M.



Then $\triangle BHK$ is the triangle required.

For, in the right triangle PEG, we have,

$$\overline{PG}^2 + \overline{GE}^2 = \overline{PE}^2,$$

or, since $PG = PD$, and $GE = FH$,

$$\overline{PD}^2 + \overline{FH}^2 = \overline{PE}^2.$$

Now, since the triangles PDK, FHM, and PEM are similar, and therefore proportional to the squares of their homologous sides, it follows that

$$\triangle PDK + \triangle FHM = \triangle PEM.$$

Adding the polygon $BFMPD$ to both sides of the above equivalence we get

$$\text{Area } BHK = \text{Area } BFED.$$

Remark. There will, in general, be two different solutions according as the side of the parallelogram drawn through P is taken parallel to BA or BC. If, however, the given area is such that $PE = PD$, the two solutions will be equal, and either solution will, in this case, give the *smallest triangle* that can be drawn with its side passing through P. If the given area is such that $PE < PD$, there will be no solution.

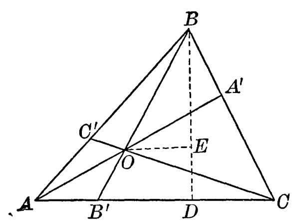
450. Proposed by W. L. WATSON, Moundsville, W. Va.

If three straight lines AA', BB', CC', drawn from the vertices of a triangle ABC to the opposite sides, pass through a common point O within the triangle, then

$$\frac{OA}{AA'} + \frac{OB}{BB'} + \frac{OC}{CC'} = 1.$$

SOLUTION BY MARCUS SKARSTEDT, Augustana College, Rock Island, Ill.

Draw BD perpendicular to AC, and OE perpendicular to DB. Then



$$\frac{OB'}{BB'} = \frac{ED}{BD} = \frac{\triangle AOC}{\triangle ABC}.$$

Similarly,

$$\frac{OA'}{AA'} = \frac{\triangle COB}{\triangle ABC} \quad \text{and} \quad \frac{OC'}{CC'} = \frac{\triangle ABO}{\triangle ABC}.$$

Adding, we get

$$\frac{OA'}{AA'} + \frac{OB'}{BB'} + \frac{OC'}{CC'} = \frac{\triangle COB + \triangle AOC + \triangle ABO}{\triangle ABC} = 1.$$

Solved similarly by A. M. HARDING, NATHAN ALTSHILLER, R. M. MATHEWS, A. H. HOLMES, T. DANTZIG, PAUL CAPRON, E. E. WHITFORD, HORACE OLSON, CLIFFORD N. MILLS, A. L. MCCARTY, and GEORGE Y. SOSNOW.

451. Proposed by CLIFFORD N. MILLS, So. Dakota State College.

Determine the sides of an isosceles triangle of given area, having given that the sum of its sides is equal to the sum of its base and altitude.

SOLUTION BY ELIZABETH BROWN DAVIS, U. S. Naval Observatory.

Let BCD be the given isosceles triangle; A, its area; $2b$ its base; $2a$ its altitude; and c each of its equal sides. Then by the conditions of the problem

$$2c = 2a + 2b, \qquad \text{or} \qquad c = a + b.$$

Also,

$$(2a)^2 + b^2 = c^2 = (a + b)^2;$$

whence

$$3a^2 + 2ab = A,$$

and $a = \frac{1}{3}\sqrt{3A}$. Hence, $2a = \frac{2}{3}\sqrt{3A} =$ altitude. Since $2ab = 3a^2$, $b = \frac{3}{2}a = \frac{1}{2}\sqrt{3A}$; $2b = \sqrt{3A} =$ the base; and $c = a + b = \frac{5}{6}\sqrt{3A} =$ the length of the equal sides.

Also solved by C. E. GITHENS, ELBERT H. CLARKE, A. M. HARDING, A. H. HOLMES, WALTER C. EELLS, H. C. FEEMSTER, HORACE OLSON, GEORGE Y. SOSNOW, and NATHAN ALTSHILLER.

CALCULUS.

364. Proposed by EMMA GIBSON, Drury College.

Solve the differential equation

$$(xp - y)^2 = a(1 + p^2)(x^2 + y^2)^{3/2}, \text{ where } p = \frac{dy}{ax}.$$

I. SOLUTION BY GEO. W. HARTWELL, Hamline University.

Let $v = \dfrac{y}{x}$ and $u^2 = x^2 + y^2$.

The equation then takes such form that the variables can be separated and we have

$$\frac{dv}{1 + v^2} = \frac{\sqrt{a}\, du}{\sqrt{u - au^2}}.$$

Integrating,

$$\tan^{-1} v + c = \cos^{-1}(1 - 2au) = \text{vers}^{-1}\, 2au.$$

$$\therefore\ \tan^{-1}\frac{y}{x} + c = \text{vers}^{-1}\, 2a\sqrt{x^2 + y^2}.$$

II. SOLUTION BY C. C. STECK, New Hampshire College, Durham, N. H.

If we put $x = r\cos\theta$ and $y = r\sin\theta$ in the given equation we get

$$d\theta = \frac{adr}{\sqrt{ar - a^2r^2}}.$$

Integrating this we have

$$\theta + c = \text{arc vers } 2ar.$$

Whence,

$$\text{arc tan}\frac{y}{x} + c = \text{arc vers } 2a\sqrt{x^2 + y^2}.$$

Solved similarly by A. M. HARDING, C. N. SCHMALL, ELMER SCHUYLER, and LEROY COFFIN.

365. Proposed by C. N. SCHMALL, New York City.

Show that the area inclosed by each of the following three curves is equal to the circle of radius a; viz., πa^2.

(1) $a^2x^2 = y^3(2a - y)$, (2) $a^2 - x^2 = (y - mx^2)^2$, (3) $(xy + c + bx^2)^2 = x^2(a^2 - x^2)$.

I. SOLUTION BY A. M. HARDING, University of Arkansas.

If we change these equations to parametric forms we obtain

(1) $x = 4a\cos^3 t \sin t, \quad y = 2a\cos^2 t,$

(2) $x = a\sin t, \quad y = ma^2\sin^2 t + a\cos t,$

(3) $x = a\sin t, \quad y = \dfrac{a^2\cos t - a^2b\sin t - c\csc t}{a}.$

Hence,

(1) $$\text{Area} = \int y\,dx = \int_0^\pi 8a^2(4\cos^6 t - 3\cos^4 t)dt = \pi a^2;$$

(2) $$\text{Area} = \int y\,dx = \int_0^{2\pi} (ma^2\sin^2 t + a\cos t)a\cos t\,dt = \pi a^2; \text{ and}$$

(3) $$\text{Area} = \int y\,dx = \int_0^{2\pi} (a^2\cos t - a^2b\sin t - c\csc t)\cos t\,dt = \pi a^2.$$

II. SOLUTION BY H. C. FEEMSTER, York College, Nebraska.

$$A_1 = \int_0^{2a}\int_{-(y/a)\sqrt{2ay-y^2}}^{(y/a)\sqrt{2ay-y^2}} dydx = \left[-\frac{3a^2+ax-2x^2}{3}\sqrt{2ax-x^2} + a^2\operatorname{vers}^{-1}\frac{x}{a}\right]_0^{2a} = \pi a^2;$$

$$A_2 = \int_{-a}^{a}\int_{mx-\sqrt{a^2-x^2}}^{mx+\sqrt{a^2-x^2}} dxdy = \left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right]_{-a}^{a} = \pi a^2;$$

and

$$A_3 = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}-(c/x)-bx}^{\sqrt{a^2-x^2}-(c/x)-bx} dxdy = \left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right]_{-a}^{a} = \pi a^2.$$

NUMBER THEORY.

218. Proposed by ELIJAH SWIFT, University of Vermont.

If p is prime > 3 show that

$$\sum_{a=1}^{a=p-1}\frac{1}{a^2} \equiv 0 \pmod{p}. \tag{1}$$

I. SOLUTION BY TRACY A. PIERCE, Berkeley, Cal.

In (1), we may replace 1 by a^{p-1}, since $a^{p-1} \equiv 1 \pmod{p}$. We then have

$$\sum_{a=1}^{a=p-1} a^{p-3} \equiv 0 \pmod{p}.$$

But it is well known that the sum of like powers of the numbers 1, 2, 3, $\cdots$, $p - 1$ is divisible by p if the power is not a multiple of $p - 1$; hence, the theorem is proved.

As a generalization of the congruence above, we may state

$$\sum_{a=1}^{a=p-1} \frac{1}{a^k} \equiv 0 \pmod{p}$$

if k is not a multiple of $p - 1$.

II. Solution by the Proposer.

$$\sum_{a=1}^{a=p-1} \frac{1}{a^2} \equiv \sum_{a=1}^{a=p-1} a^2 \pmod{p}.$$

But

$$\sum_{a=1}^{a=p-1} a^2 \equiv A_1^2 - 2A_2$$

where

$$A_1 = \sum_{a=1}^{p-1} a, \qquad A_2 = \sum_{\substack{a_1=1\\a_2=1}}^{p-1} a_1 a_2, \qquad a_1 \neq a_2.$$

In Bachmann's *Niedere Zahlentheorie*, Vol. I, page 155, it is proved that A_1 and A_2 are divisible by p, whence the theorem.

NOTES AND NEWS.

EDITED BY W. D. CAIRNS.

Professor ANDREW W. PHILLIPS, of Yale University, died on January 20, 1915. He was joint author of Phillips and Fisher's *Geometry.*

Professor R. M. BARTON of the University of New Mexico has been appointed professor of mathematics in Lombard College.

Professor W. H. ROEVER, representative of Washington University on the editorial staff of the MONTHLY, was elected to honorary membership in the Society of Phi Beta Kappa by the Washington University Chapter, which was installed last year.

"The arithmetic mean as approximately the most probable value *a posteriori* under the gaussian probability law," by EDWARD L. DODD, is the title of a pamphlet published by the University of Texas as its January, 1915, *Bulletin.*

The February number of the *Proceedings of the National Academy of Sciences* contains two mathematical articles. The first is by Professor E. J. WILCZYNSKI and bears the title "Conjugate systems of space curves." The second is due to

Professor L. P. EISENHART and appeared under the heading "Transformation of surfaces omega."

The *American School Board Journal* for December prints an article by Mr. GEORGE H. ECKELS on "The place of mathematics in the highschool curriculum."

Professor A. O. LEUSCHNER of the University of California was elected vice-president for 1915 of Section A (Mathematics and Astronomy) of the American Association for the Advancement of Science.

The central committee of the International Commission on the Teaching of Mathematics has found it necessary to abandon the meeting planned for Munich in August of this year and to postpone the preparation of such reports as concern European countries.

Professor S. W. SHATTUCK, who until his retirement in 1912 was for thirty-seven years head of the department of mathematics in the University of Illinois, died in Champaign on February 13.

The first award of the Alfred-Ackermann-Teubner Prize, which was inaugurated in 1912 for the advancement of the mathematical sciences, has been made to Professor FELIX KLEIN for his work relating to the teaching of mathematics. The money value of this prize is 1,000 marks, and it is to be awarded every two years. The subject for 1916 is "Mathematics, primarily arithmetic and algebra."

The "List of officers and members" of the American Mathematical Society published in January shows a membership at present of 709, 38 having been admitted and 32 having withdrawn during 1914. New York State furnishes the largest group, 115 members, 70 of whom are in New York City. There are 56 members from foreign countries.

Professor de la VALLÉE POUSSIN of the University of Louvain is giving a series of lectures in French at Harvard University on Lebesgue Integrals. The lectures are given twice or three times a week during the second semester and are accompanied by supplementary lectures and explanations in English by Dr. DUNHAM JACKSON.

At the University of Illinois the graduate students in mathematics recently formed a temporary organization with a view to securing lectures on mathematical topics which are not commonly treated in the regular courses. During the present semester these lectures are being given by Professors G. A. MILLER and J. B. SHAW on the general subjects, historical development and philosophy of mathematics.

In the February number of *School Review* Mr. J. H. MINNICK of Horace Mann High School, Columbia University, reports on "A comparative study of the mathematical abilities of boys and girls," based upon the work of 150 boys and

243 girls in the Bloomington (Indiana) high school during the four years beginning September, 1906. Their relative achievements in English, history, language and science were also tabulated and certain definite conclusions were reached by the author. Taking into account the whole student body, the girls are the equals of the boys although they do not excel to the same degree in mathematics as in some other subjects, especially in language and English. Among the retarded students, mathematics has given slightly more trouble to girls than to boys; mathematics is evidently a slightly stronger factor in the elimination of girls than of boys. Measured by ability to achieve, mathematics is about as well suited to girls as are history and science.

Further, the records of 191 students who entered the high school during the years 1903–1909 and later studied in Indiana University were considered, this study indicating that while smaller percentages of girls are conditioned and failed, the girls as a group do not maintain their standing in the university quite as well as do the boys.

The annual meeting of the British Mathematical Association was held at the London Day Training College, London, on January 9, under the presidency of Sir George Greenhill. It will be of interest to our readers, by way of comparison with American associations, to know that this body now enrolls 749 members, of whom 8 are honorary and 80 are life members; aside from these there are about 200 associates. Professor A. N. Whitehead was elected president for the years 1915 and 1916.

During the past year the Council has issued a *Catalogue of Current Mathematical Journals, etc.*, with the names of the libraries in which they may be found; the work was done largely through W. J. Greenstreet, editor of the *Mathematical Gazette*, the organ of the Association.

In continuance of the important work done in the past few years, two committees may be mentioned; namely, (1) a subcommittee has been considering the whole question of the teaching of geometry and is now engaged in drawing up a report; (2) a special committee is preparing a report on the teaching of mathematics in girls' schools. The latter report is to include not only a fairly detailed scheme of work in mathematics for girls, both specialists and non-specialists, but also suggestions as to methods of teaching the subject.

Sir George Greenhill's presidential address on "Mathematics in artillery science" is reported in a summarized form in *Nature* for January 21. This address has not as yet appeared in print, but has been reported only in the daily papers; a possible explanation is his severe criticisms of England's lack of attention to the theoretical grounding of military science.

The seventeenth conference of the secondary schools cooperating with the University of Chicago was held at Chicago on Friday, April 16, 1915. The general topic for all the departmental sections was on the use of the school library. In the mathematics section this topic had already been discussed at the preceding conference and hence the meeting was devoted to the consideration of two

other topics, namely, (1) the contribution of mathematical clubs to interest and activity among high school pupils, and (2) the closer interrelations that should exist between the mathematics of the sixth, seventh, and eighth grades and the mathematics of the high school, considered from the standpoint both of efficiency in teaching and of economy in time.

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THE HISTORY OF ZENO'S ARGUMENTS ON MOTION.

By FLORIAN CAJORI, Colorado College.

VI.

6. NEWTON, BERKELEY, JURIN, ROBINS AND OTHERS.

Whether certain variables can reach their limits or not is the vital issue in the "Achilles." For that reason Newton's statements on this point are of interest:

". . . those ultimate ratios with which quantities vanish are not truly the ratios of ultimate quantities, but limits toward which the ratios of quantities decreasing without limit do always converge; and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished in *infinitum*."[1]

That Newton let his variables reach their limits appears even more clearly in the following passage:

"Quantities, and the ratios of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer the one to the other than by any given difference, become ultimately equal."[2]

Other passages in the first book of the *Principia* allow variables to reach their limits. While Newton's exposition is not as explicit as one might wish, nor free from objection, he deserves the credit of perceiving that variables may reach their limits and that variables arising in mechanics are usually of such a nature that they do reach their limits.

As is well known, the foundations of the calculus were severely attacked by Bishop Berkeley. His first published statement on this subject appears in his *Alciphron, or the Minute Philosopher* (1732), penned while he and his wife were sojourning at Newport, Rhode Island. He says that mathematical science falls short of "those clear and distinct ideas" which many "expect and insist upon in the mysteries of religion. . . . Such are those which have sprung up

[1] Newton's *Principia*, Book I, Section I, last scholium.
[2] Newton's *Principia*, Book I, Section I, Lemma I.

in geometry about the nature of the angle of contact, the doctrine of proportions, of indivisibles, infinitesimals, and divers other points." He argues that, just as mathematics, though involved in obscurities, is esteemed excellent and useful, so articles of Christian faith should be accepted as none the less true and excellent, because they afford matter of controversy. These ideas Berkeley elaborates more fully in his *Analyst* (1734) and the *Defence of Free-thinking in Mathematics* (1735). These attacks on the foundations of the calculus deserve mention in this history, even though no reference is made to Zeno's arguments. Berkeley was familiar with Zeno's arguments, for in an earlier essay he refers to them twice, though without critical comment.[1]

The *Analyst* is a discourse addressed to an unnamed infidel mathematician, said to have been Dr. Halley. Berkeley's lengthy discourse dwells mainly on two points: (1) The conception of fluxions is unintelligible, since they are the ratios of quantities that have no magnitudes, (2) the derivation of the fluxion of x^n rests on a violation of an axiomatic canon of sound reasoning. Newton did not take the fluxions infinitely small, but he originally took the *moments of fluxions* to be infinitely small. Later he discarded the infinitely little and explained fluxions by his theory of prime and ultimate ratios. Berkeley argued with great acuteness against the infinitely small. As his arguments do not directly apply to the "Dichotomy" and the "Achilles" we shall not go into details except to quote part of *Query* 21 in his *Analyst*, where he inquires, "whether the supposed infinite divisibility of finite extension hath not been a snare to mathematicians and a thorn in their sides?" Before this Berkeley had discussed this vital question quite fully in his *Principles of Human Knowledge*, first printed in 1710, then again in 1734. Infinite divisibility, because inapprehensible by our senses, is dismissed from his philosophy as void of meaning or involving contradictions. We quote the following:[2]

"And, as this notion is the source from whence do spring all those amusing geometrical paradoxes which have such a direct repugnancy to the plain common sense of mankind, and are admitted with so much reluctance into a mind not yet debauched by learning. . . . Of late speculations about Infinites have run so high, and grown to such strange notions, as have occasioned no small scruples and disputes among the geometers of the present age. Some there are of great note who, not content with holding that finite lines may be divided into an infinite number of parts, do yet farther maintain that each of those infinitesimals is itself subdivisible into an infinity of other parts or infinitesimals of a second order, and so on *ad infinitum.* These, I say, assert there are infinitesimals of infinitesimals of infinitesimals, etc., without ever coming to an end; so that according to them an inch does not barely contain an infinite number of parts, but an infinity of an infinity of an infinity *ad infinitum* of parts. Others there be who hold all orders of infinitesimals below the first to be nothing at all; . . . Have we not therefore reason to conclude they are *both* in the wrong, and that there is in effect no such thing as parts infinitely small, or an infinite number of parts contained in any finite quantity? . . . If it be said that several theorems undoubtedly true are discovered by methods in which infinitesimals are made use of, which could never have been if their existence included a contradiction in it—I answer that upon a thorough examination it will not be found that in any instance it is necessary to make use of or conceive infinitesimal parts of finite lines, or even quantities less than the *minimum sensible;*

[1] A. C. Fraser, *The Works of George Berkeley*, D.D., Vol. II, Oxford, 1871, p. 501, also Vol. III, pp. 76, 91.

[2] Berkeley, "Principles of Human Knowledge," *The Works of George Berkeley*, Vol. I, pp. 220, 223–225.

nay, it will be evident this is never done, it being impossible. And, whatever mathematicians may think of fluxions, or the differential calculus and the like, a little reflection will show them that, in working by those methods, they do not conceive or imagine lines or surfaces less than what are perceivable to sense."

The *Analyst* gave rise to a spirited discussion. An anonymous reply by Philalethes Cantabrigiensis appeared under the title, *Geometry, no Friend to Infidelity; or, A Defence of Sir Isaac Newton and the British Mathematicians,* London, 1734. The authorship of this letter has been attributed to Conyers Middleton and Robert Smith, but George A. Gibson makes it plain that the author is James Jurin,[1] a noted Cambridge physician and an admirer of Newtonian philosophy, which he had imbibed from Newton himself. Philalethes admits that the doctrine of fluxions is involved in difficulties, but claims that they are not insuperable. Gibson calls this reply "an extremely weak defence" of the doctrine of fluxions. In 1735 Berkeley published his *Defence, etc.,* alluded to above, to which Philalethes replied in a pamphlet entitled *The Minute Mathematician: or the Freethinker no just Thinker.* Berkeley did not make answer to this, nor to a publication of the same year by Benjamin Robins, a mathematician and military engineer, which appeared in London under the title: *A Discourse concerning the Nature and Certainty of Sir Isaac Newton's Methods of Fluxions and of Prime and Ultimate Ratios.* A controversy arose between Philalethes and Robins which bears more closely on our present topic than that between Philalethes and Berkeley. Robins and Philalethes differed in the interpretation of Newton; they "began that long struggle in which," as Gibson puts it, "Robins proved his immense superiority to his antagonist, alike in temper, in general mathematical learning, and in special knowledge of Newton's fluxionary methods." The part of the debate which interests us just now relates to the variable's reaching its limit. On this point Robins's vision was somewhat circumscribed; he held that no variable could possibly reach its limit. This interpretation of Newton is at variance with that usually accepted. For the purposes of debate it was no doubt easier for Robins to limit himself to variables which do not reach their limits; from the standpoint of mathematical theory which should be broad enough to explain all ordinary phenomena of motion, his position was unfortunate. Says Robins:

"It was urged that the quantities or ratios, asserted in this method to be ultimately equal, were frequently such as could never absolutely coincide. As, for instance, the parallelograms inscribed within the curve, in the second *lemma* of the first book of Sir Isaac Newton's *Principia,* cannot by any division be made equal to the curvilinear space they are inscribed in, whereas in that *lemma* it is asserted that they are ultimately equal to that space.

"Here two different methods of explanation have been given. The first, supposing that by ultimate equality a real assignable coincidence is intended, asserts that these parallelograms and the curvilinear space do become actually, perfectly, and absolutely equal to each other."

This last view described by Robins was the view of Jurin. No doubt Jurin followed more nearly in Newton's footsteps than did Robins. Newton declares

[1] G. A. Gibson, Review of Cantor's "Geschichte der Mathematik," Vol. 3, in *Proceedings of the Edinburgh Mathematical Society,* Vol. XVII, 1898–99. Gibson's article gives the most complete account of the *Analyst* controversy with which we are familiar.

that the variable becomes "ultimately equal" to its limit, yet Robins insists that he must have seen they would always remain unequal. Robins's contention was hardly valid; whether a variable reaches its limit or not depends wholly upon the variable. Now a law of variation may be artificially established by the human mind. That law may be such that the variable reaches its limit, or it may be such that the variable does not reach its limit. Apparently Robins, perhaps unconsciously, assumed laws of variation which kept the variable and its limit constantly apart, while the great Newton conceived modes of variability not limited to such conditions. How Robins came to insist that his views were those deduced from Newton's *Principia* is elucidated by him in the following passage, in which he says:

"[Newton] has given such an interpretation of this method as did no ways require any such coincidence. In his explication of that doctrine of prime and ultimate ratios he defines the ultimate magnitude of any varying quantity to be the limit of that varying quantity which it can approach within any degree of nearness, and yet can never pass. And in like manner the ultimate ratio is the limit of that varying ratio."

The reader may compare this passage from Robins with the passages in Newton's *Principia* which we quoted earlier. That Philalethes failed to do Newton justice is clearly brought out by Gibson. According to Philalethes an ultimate ratio is not the *limit* of a varying ratio, but the last value of a ratio. Berkeley very properly argued that there is no last value of the augments except zero, so that the phrase "the ratio with which they vanish," used by Newton himself, does not represent any mathematical operation, and itself requires explanation. Gibson claims that Newton's terminology of first and last ratios was unfortunate, "as it lent itself too readily to an interpretation in the sense of indivisibles; and it was this interpretation that Berkeley and Philalethes alike proceeded upon. Were that interpretation correct, then Berkeley's contentions would in the main be fully justified."

We may sum up the discussion by saying that Berkeley did not directly inquire whether Achilles caught the tortoise or not; that according to the teachings of Newton and Jurin on limits, Achilles did catch the tortoise, though it is not quite evident how the feat was accomplished; that Robins's theory did not allow Achilles to overtake the tortoise, though Achilles would come tantalizingly near doing so.

It was in 1710 that Bayle's famous dictionary was translated from French into English. We are not able to trace any immediate influence of the article on "Zeno of Elea" upon English thought. In 1713 appeared the *Clavis Universalis* of the English divine, Arthur Collier, an idealist who aimed to prove in his book the non-existence of the external world. As his fifth argument he considers motion. He does not mention Zeno, nor any other philosopher, but deserves to rank among Zeno's boldest and most reckless disciples. A few quotations will suffice.[1]

"A world, in which it is both possible and impossible that there should be any such thing as motion, is not at all;

[1] A. Collier, *Clavis Universalis*, edited by Ethel Bowman, Chicago, 1909, pp. 80–82.

But this is the case of an external world;
Ergo, there is no such world."

". . . Now in such translation the space or line through which the body moved is supposed to pass, must be actually divided into all its parts. This is supposed in the very idea of motion; but this all is infinite, and this infinite is absurd, and consequently it is equally so, that there should be any motion in an external world."

". . . to affirm that a line by motion or otherwise is divided into infinite parts, is in my opinion to say all the absurdities in the world at once. For, *first*, this supposes a number actually infinite, that is, a number to which no unit can be added, which is a number of which there is no sum total, that is, no number at all; consequently, *secondly*, by this means the shortest motion becomes equal to the longest, since a motion to which nothing can be added must needs be as long as possible. This also, *thirdly*, will make all motions equal in swiftness, it being impossible for the swiftest in any stated time to do more than pass through infinite points, which yet the shortest is supposed to do."

Collier gives no evidence of having looked into the higher mathematics as did Berkeley and Hume. After referring to the angle of contact between a circle and its tangent, which "is infinitely less than any rectilineal angle," Hume concludes "that all the ideas of quantity upon which mathematicians reason, are nothing but particular, and such as are suggested by the senses and imagination and consequently, cannot be infinitely divisible."[1]

Zeno's arguments appear to have been discussed but little in England during the second half of the eighteenth century. Charles Hutton refers to them in the article "Zeno, Eleates" in his *Mathematical and Philosophical Dictionary*, London, 1795. He describes the "Achilles" and remarks that "the fallacy will soon be detected," as the time can easily be computed when Achilles will not only have overtaken, but actually passed the tortoise.

Some English mathematicians kept in the path laid out by Newton, by teaching that a variable reaches its limit. Especially is this true of mathematicians at Cambridge, from Jurin to Whewell and from Whewell to Todhunter. Says Whewell:[2] "A magnitude is said to be *ultimately equal* to its Limit; and the two are said to be *ultimately in a ratio of equality.*"

Hutton, who was professor of mathematics at the Royal Military Academy, Woolwich, says, in his *Dictionary*, under the word "Limit," that the variable "can never go beyond it." A different exposition was given by Augustus De Morgan. In the article "Progressions" in the *Penny Cyclopædia* (London, 1841) he says of Achilles: "Let him go as far as he may, he must always come up to where the tortoise was before he can reach the point; so that it requires *an infinite number of parts of time*, but here the sophism quietly introduces *an infinite time* to catch the tortoise." De Morgan establishes the two convergent series, the one for the time, the other for the distance, passed over by Achilles, but he ignores the crucial question as to the reaching of the limit. In the article "Limit" he says that the variable "must never become equal" to its limit. Consequently De Morgan's exposition of limits, as given in these articles, lacked the generality necessary to explain the "Achilles."

On the Continent there prevailed the same diversity of definitions of a limit.

[1] D. Hume, *Essays Moral, Political, and Literary*, London, 1898, edited by T. H. Green and T. H. Grose, Vol. II, p. 129.

[2] William Whewell, *Doctrine of Limits*, Cambridge, 1838, p. 18. See also p. 23.

D'Alembert in 1754 puts no restriction upon the variable reaching its limit;[1] only, the variable must not "surpasser la grandeur dont elle approche." It is well known that there was a time when Lagrange was greatly troubled by the lack of rigor in the foundations of the calculus. He said:

> "That method [of limits] has the great inconvenience of considering quantities in the state in which they cease, so to speak, to be quantities; for though we can always well conceive the ratio of two quantities, as long as they remain finite, that ratio offers to the mind no clear and precise idea, as soon as its terms become, the one and the other, nothing at the same time."[2]

In the nineteenth century Carnot[3] and Cauchy[4] put no restriction upon variables reaching their limits. In 1817 Bolzano, whose writings did not at the time receive the attention they deserved, was concerned with the limits of continuous functions which attain their limits.[5] Later some French writers thought it necessary to impose restrictions. With Duhamel[6] the variable "never reaches" its limit. In Germany Klügel[7] gives a definition placing no restriction, but in the comments which follow the variable is pictured as not reaching its limit. In 1871 Hermann Hankel starts out in his article "Grenze"[8] by defining what is called a limit in mathematics; the limit is not reached. The difference between the variable and its limit he calls an infinitely small quantity—a quantity no multiple of which is capable of producing unity. But magnitudes of the same kind, by Euclid V, Def. 4, are such that some multiple of one will exceed the other. Hence an infinitely small line is not of the same kind as a finite one. This contradiction is to Hankel one of the indications that a scientific treatment of limits is still wanting. Hankel proceeds to express his adherence to the actual infinite and to develop a more satisfactory definition, free from restriction as to the attainment of the limiting value.

In the United States, as elsewhere, there has been great diversity of practice. Charles Davies of West Point, later of Columbia College, lets the variable reach its limit, in his *Calculus* of 1836. A discussion of this subject was carried on in the *Analyst* by Levi W. Meech, C. H. Judson, De Volson Wood and Simon Newcomb.[9] Wood's article voices the view that prohibiting the variable from attaining its limit "unnecessarily restricts the law of approach of the variable," though the variable can be "subjected to such a law that to the human mind it will appear impossible for it to reach the limit." An elaborate discussion of the

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[1] Article "Limite" in the *Encyclopédie, ou Dictionnaire raisonné des sciences* (Diderot).

[2] Quoted by Bledsoe, and by Carnot in his *Reflexions sur la métaphysique du calcul infinitesimal*, 5. éd., Paris, 1881, p. 147.

[3] Carnot, *op. cit.*, p. 168.

[4] A. L. Cauchy, *Cours d'analyse*, 1821, p. 4.

[5] Philip E. B. Jourdain, "The Development of the Theory of Transfinite Number," *Archiv der Mathematik u. Physik*, Bd. 14, 1909, p. 297. Jourdain's work appears in Bd. 10, 1906, pp. 254–281; Bd. 14, 1909, pp. 289–316; Bd. 16, 1910, pp. 21–43; Bd. 22, 1913.

[6] *Eléments de calcul infinitesimal*, Duhamel, Vol. I, Book. I, Chap. 1.

[7] G. S. Klügel, Mathematisches Wörterbuch, Leipzig, 1805, Vol. II, Art. "Grenze."

[8] *Allgem. Encyklopädie der Wissensch. u. Künste* (Brockhaus), 90. Theil.

[9] *The Analyst* (J. E. Hendricks, Des Moines, Iowa), Vol. I, 1874, p. 133 et seq.; Vol. VIII, 1881, p. 105 et seq.; Vol. IX, 1882, p. 79 et seq.; Vol. IX, 1882, p. 114 et seq.

subject is found in A. T. Bledsoe's *Philosophy of Mathematics*, Philadelphia, 1886. He holds (p. 44) that the variable never actually attains its limit, and

". . . this, I apprehend, will be found to be the case in relation to every variable really used in the infinitesimal method. It will, at least, be time enough to depart from the definition of Duhamel when variables are produced from the calculus which are seen to reach their limit without violating the law of their increase or decrease."

That a teacher who had pondered so long upon the foundations of the calculus as Bledsoe had done, could not think of examples of variables reaching their limits is an indication that the application of the calculus to physics and mechanics did not then receive the careful attention it deserved.

It is with the theory of limits as with negative numbers and imaginaries. In the eighteenth century it was felt that, whether such numbers could exist in algebra, was a matter of argument and demonstration; now it is merely a question of assumption. The same is true with variables reaching their limits. In modern theory it is not particularly a question of argument, but rather of assumption. The variable reaches its limit if we will that it shall; it does not reach its limit, if we will that it shall not. Our "willing" the one thing or the other consists in assuming a continuum in which the limit is a value the variable can assume; our "not willing" consists in not assuming, in the aggregate of values the variable can take, the value of the limit.

A GENERAL FORMULA FOR THE VALUATION OF BONDS.

By C. H. FORSYTH, University of Michigan.

It is the purpose of the present paper to generalize a formula for the valuation of bonds so that it will be applicable to a large number of bond offerings not satisfactorily covered by a known formula.

The problem to be considered may be put more concretely as follows. All formulas known up to the present time, cover the offering of bonds or loans only where the principal is repaid in *equal* installments. The case where payment of the principal is made in a lump sum is included, as a special case.

We shall derive a formula for computing the price of bonds where this principal is repaid in general in *unequal* installments. This formula will include cases where there may be only one installment of any one value. In fact, the formula will include as special cases not only the most general formula known at present but also all the special cases of the latter.

The most general formula[1] for valuation of bonds known up to this time is

$$k = \left(1 - \frac{a_{\overline{m(f+tr)}|} - a_{\overline{mf}|}}{r a_{\overline{mt}|}}\right)\left(\frac{g-i}{i}\right), \tag{1}$$

where k represents the premium or discount—as the case may be.

The nature of the above bond offering or loan is as follows. The principal

[1] J. W. Glover, *A general formula for evaluating securities*, this MONTHLY, March, 1915.

of one dollar is to be repaid in r *equal* installments, the first at the end of f years and the rest at intervals of t years. The interest named in the bond offering is g and the interest rate to be actually realized, and hence used as a basis of the annuities of the formula itself, is i, both rates to be paid m times a year. It is unnecessary to give the derivation of formula (1) here.

Formula (1) is rarely used in the form given, much simpler formulas—special cases of (1)—being ordinarily used; but as mentioned above, it covers only those bond offerings where the repayments of the principal are *equal*. If a problem however simple, arises where the repayments are *unequal*, it is necessary to consider as many distinct offerings as there are different installments or repayments.

We propose now to derive a formula, using formula (1) as a basis, to apply to cases where these installments are, in general (but not necessarily), *unequal*.

The nature of such a bond is as follows. The principal of S dollars is to be repaid in $r_1 + r_2 + \cdots + r_n$ installments, of which the first r_1, the next r_2, etc., are equal, there being n *different* installments.

The first installment is to be paid at the end of f_1 years, the next $r_1 - 1$ at intervals of t_1 years, the first of the r_2 payments at the end of f_2 years (from the date of the whole bond offering), the next $r_2 - 1$ at intervals of t_2 years, etc., and if we designate as " major intervals " those intervals wherein all the installments are equal, we may say in general that in the nth major interval there will be r_n installments at intervals of t_n years, the first of which is to be paid at the end of f_n years.

The rate of interest offered in the loan is g and the rate to be actually realized is i, both to be paid m times a year.

We shall let S_1 represent the sum of the r_1 equal installments of the first major interval, and in general S_n the sum of the r_n equal installments of the nth major interval; and we shall also let I_n represent the value of each of these installments, that is put I_n equal to S_n/r_n.

Then applying formula (1) itself, considering the several major intervals as so many different bond offerings, the value of the premium or discount on the whole bond offering becomes at once

$$\begin{aligned} k = {} & \frac{S_1}{S}\left(1 - \frac{a_{\overline{m(f_1+r_1t_1)}|} - a_{\overline{mf_1}|}}{r_1 a_{\overline{mt_1}|}}\right)\left(\frac{g-i}{i}\right) \\ & + \frac{S_2}{S}\left(1 - \frac{a_{\overline{m(f_2+r_2t_2)}|} - a_{\overline{mf_2}|}}{r_2 a_{\overline{mt_2}|}}\right)\left(\frac{g-i}{i}\right) \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + \frac{S_n}{S}\left(1 - \frac{a_{\overline{m(f_n+r_nt_n)}|} - a_{\overline{mf_n}|}}{r_n a_{\overline{mt_n}|}}\right)\left(\frac{g-i}{i}\right). \end{aligned}$$

Adding, we have

$$k = \left(1 - \frac{\sum_1^n a_n}{S}\right)\left(\frac{g-i}{i}\right), \tag{2}$$

where we let $A_n = I_n/a_{\overline{mt_n}|}$ and $a_n = A_n(a_{\overline{m(f_n+r_nt_n)}|} - a_{\overline{mf_n}|})$.

Ordinarily we might expect the first interval to be the same as the other intervals of any major interval. However, it should be emphasized that nothing has been assumed to prevent this first interval from being any number of years. In other words, we may, if we choose, look upon the first installment of each major interval as being deferred any arbitrary number of years just as it is openly assumed in the first major interval. This period of deferment may, of course, take as a special case the value of each of the other intervals (t_n in the nth major interval) of that major interval. If, in this special case, we have also $t_1 = t_2 = \cdots = t_n = t$, then

$$f_1 = f,\ f_2 = f + r_1 t,\ f_3 = f + (r_1 + r_2)t,\ \cdots,\ f_n = f + (r_1 + r_2 + \cdots + r_{n-1})t$$

and formula (2) becomes

$$k = \left(1 - \frac{\sum_0^n (I_n - I_{n+1})a(n)}{Sa_{\overline{mt}|}}\right)\left(\frac{g-i}{i}\right), \tag{3}$$

where $a(n) = a_{\overline{m[f+(r_1+r_2+\cdots+r_n)t]}|}$, $a(0) = a_{\overline{mf}|}$, and $I_0 = I_{n+1} = 0$.

Formula (3) appears more complicated than (2) but on practical application will prove much simpler and, of course, much less general. One must be careful in assigning the proper values to the f's in a practical problem in using formula (2). In general, f_n extends from the date of the whole bond offering till the first installment of the nth major interval.

As an example and special case where we have only two major intervals or only two different installments, formula (3) may be expanded into the form

$$k = \left(1 - \frac{a(2)I_2 + a(1)(I_1 - I_2) - a(0)I_1}{Sa_{\overline{mt}|}}\right)\left(\frac{g-i}{i}\right), \tag{4}$$

and so on, in general, for n equal to any positive integer.

Perhaps we should add that in case any major interval should contain only one installment (one of the r's becomes unity) it may be asked what value should be given the corresponding value of t for that major interval. On investigation one will find that in that case the troublesome terms involving that particular t will cancel out in the formula and hence prove immaterial in value in practice. However, we must be careful to not give it the value zero, for in that one case an indeterminate expression will arise and make unnecessary trouble. We would suggest that the simplest plan in practice would be to assume the value of that particular t to be unity, as that value would make a minimum of trouble and will not necessitate the previous and troublesome cancellation of the terms in t.

If we assume n to be unity or only one value for the values of the installments, our formula (3) reduces to

$$k = \left(1 - \frac{a(1)I_1 - a(0)I_1}{Sa_{\overline{mt}|}}\right)\left(\frac{g-i}{i}\right),$$

which is identically the well-known formula (1).

It should be added that all annuities are to be valued, as formerly with formula (1), at the rate of interest i/m. Formulas (2) and (3) prove very easy to apply in a concrete problem and the values of the annuities involved are easily determined. We shall conclude with a typical though simple example. Let it be desired to find the value or price of a loan of \$50,000 offered at 6 per cent. and dated September 1, 1914; the principal to be repaid in fifteen installments, \$2,000 September 1, each year from 1919 to 1928 and \$6,000 September 1, from 1929 to 1933, to net the purchaser 5 per cent.

Here we have $f = 5$, $m = t = 1$, $g = .06$, $i = .05$ and $n = 2$ and we can use formula (3) as simplified and expressed in formula (4), whence

$$k = \left(1 - \frac{a(2) \cdot 6000 + a(1)(2000 - 6000) - a(0) \cdot 2000}{50{,}000 a_{\overline{1}|}}\right)\left(\frac{.06 - .05}{.05}\right),$$

where

$$a(2) = a_{\overline{m(f+r_1t+r_2t)}|} = a_{\overline{20}|}$$

$$a(1) = a_{\overline{m(f+r_1t)}|} = a_{\overline{15}|}$$

$$a(0) = a_{\overline{mf}|} = a_{\overline{5}|}, \text{ all to be valued at 5 per cent.}$$

The values of these annuities can be obtained from any set of tables. Hence, finally, the premium is $k = .0966977$, and the price of the entire loan becomes \$50,000 $(1 + k)$ or \$54,834.89.

A GEOMETRICAL INTERPRETATION OF GREEN'S FORMULA.

By W. V. LOVITT, Purdue University.

It is the object of this note to give a geometrical interpretation of Green's formula:

$$\int_{(C)} P(x, y)dx + Q(x, y)dy = \int_A\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

The single integral is to be taken in the positive direction around a closed curve

$$(C) \qquad C(x, y) = 0$$

in the xy-plane and the double integral over the interior A of C. To derive this formula it is sufficient to prove the formulas

$$(1) \qquad \int_A\int \frac{\partial P}{\partial y} dxdy = -\int_{(C)} Pdx,$$

$$(2) \qquad \int_A\int \frac{\partial Q}{\partial x} dxdy = \int_{(C)} Qdy$$

(see for example Goursat-Hedrick, § 126), and for simplicity it may be assumed

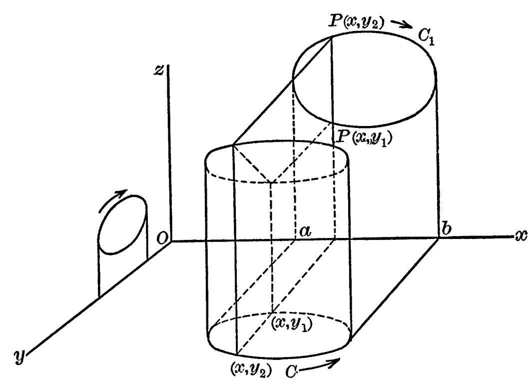
that A is a region of the xy-plane in which $\partial P/\partial y$ and $\partial Q/\partial x$ are of constant sign. We will discuss equation (1) and leave equation (2) for the consideration of the reader.

Construct the surface

$$(P) \qquad z = P(x, y).$$

The cylinder $C(x, y) = 0$ intersects the surface (P) in a curve the projection of which on the xz-plane we designate by C_1. Let the two parts of C be represented by

$$y = y_1(x), \qquad y = y_2(x)$$



as in the figure. Then the two parts of C_1 are

$$z = P(x, y_1), \qquad z = P(x, y_2)$$

and it is clear that the integral

$$\int_{(C)} P dx = \int_{(C_1)} z dx$$

represents the area of C_1. The double integral may be written

$$\int_A\int \frac{\partial P}{\partial y} dxdy = \int_A\int \frac{\dfrac{\partial P}{\partial y}}{\sqrt{\left(\dfrac{\partial P}{\partial x}\right)^2 + \left(\dfrac{\partial P}{\partial y}\right)^2 + 1}} \sqrt{\left(\frac{\partial P}{\partial x}\right)^2 + \left(\frac{\partial P}{\partial y}\right)^2 + 1} dxdy$$

$$= \int_A\int \cos \beta dS,$$

where dS is the element of the surface (P) and β is the angle between the normal to (P) and the y-axis. This shows that the double integral is also an expression for the area of C_1. It is now evident that the two sides of the equation (1) have the same absolute value. There remains the question of sign.

A curve in the xy-plane will be said to be traversed in the positive sense when it is gone over in the direction indicated by the arrow in the figure. For the xz-plane the positive direction is taken as opposite to that indicated by the arrow. For the yz-plane it will be found necessary in the discussion of equation (2) to take the positive direction as indicated by the arrow in the figure. Areas are to be considered positive when their boundaries are traversed in the positive sense and negative when their boundaries are traversed in the negative sense. These definitions of the positive sense for the three coordinate planes are independent of the octants and seem to be the simplest for our present purpose.

By an inspection of the figure we see that the curves C and C_1 are traced by corresponding points in the same sense when $\partial P/\partial y < 0$ and in opposite senses when $\partial P/\partial y > 0$. If then we take into account the sign of the projective factor $\partial P/\partial y$ and the sense in which the curve C_1 is traversed when the curve C is gone over in the positive sense, we find that always

$$-\int_{(C)} Pdx = \int_A\int \frac{\partial P}{\partial y}\,dxdy.$$

ON A CERTAIN CLASS OF DETERMINANTS.

By ERNESTO PASCAL, Milan, Italy.

Translated by permission from the *Rendiconti della R. Accademia delle Scienze Fisiche e Matematiche di Napoli* (3), volume XX, 1914.

The AMERICAN MATHEMATICAL MONTHLY for June, 1914, volume 21, page 184, contains the following question under the heading "A simple algebraic paradox."

Given two linear homogeneous complex equations

$$(1)\qquad \begin{aligned}(a+bi)(p+qi)+(c+di)(r+si)&=0,\\ (a'+b'i)(p'+q'i)+(c'+d'i)(r'+s'i)&=0.\end{aligned}$$

In order that these should be compatible it is necessary and sufficient that

$$\begin{vmatrix}(a+bi) & (c+di)\\ (a'+b'i) & (c'+d'i)\end{vmatrix}=0.$$

This complex equation is equivalent to the *two* real equations

$$(1)\qquad \begin{aligned}ac'+a'c&=bd'-b'd,\\ ad'+bc'&=a'd+b'c.\end{aligned}$$

Both of these must be fulfilled if (1) is to subsist. On the other hand equations (1) are equivalent to

$$\begin{aligned}ap-bq+cr-ds&=0,\\ bp+aq+dr+cs&=0,\\ a'p-b'q+c'r-d's&=0,\\ b'p+a'q+d'r+c's&=0.\end{aligned}$$

These are compatible when, and only when, a *single* equation is satisfied, namely

$$(2)\qquad \begin{vmatrix} a & -b & c & -d \\ b & a & d & c \\ a' & -b' & c' & -d' \\ b' & a' & d' & c' \end{vmatrix} = 0.$$

Which answer is right?

How can a single condition be equivalent to two? This is the question asked by the American mathematician.

The answer is simple. The determinant (2) may be expressed as a sum of two squares, more specifically as the sum of the squares of the two right members of (1). Consequently the vanishing of (2) implies both of the equations (1).

But this remark suggests immediately a whole class of determinants (with real elements) *which have the remarkable property of being expressible as a sum of two squares.* The law of formation of these determinants is similar to that of the so-called skew symmetric determinants, with this difference however that the property of skew symmetry does not hold with respect to the elements of the determinant themselves, but with respect to certain partial matrices included in the determinant. The following determinant is representative of this type.

$$D = \begin{vmatrix} a_1' & \cdots & a_n' & b_1' & \cdots & b_n' \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1^{(n)} & \cdots & a_n^{(n)} & b_1^{(n)} & \cdots & b_n^{(n)} \\ -b_1' & \cdots & -b_n' & a_1' & \cdots & a_n' \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -b_1^{(n)} & \cdots & -b_n^{(n)} & a_1^{(n)} & \cdots & a_n^{(n)} \end{vmatrix}.$$

Such a determinant would be obtained in connection with a system of n linear homogeneous equations of the kind considered at the beginning of this note, by carrying out the discussion there given in analogous fashion.

Since the condition $D = 0$ must correspond to two relations, one might conclude *indirectly* that *D must be a sum of two squares.* But this fact may also be proved directly by the following method, in expounding which we shall confine ourselves to the case $n = 3$. The same method however is applicable for any value of n.

If we expand the determinant D (for $n = 3$) as a sum of products of the minors contained in the first n columns by their algebraic complements, and make use of the customary notation of the symbolic theory of n-ary forms, we find

$$\begin{aligned} D = {} & (a'a''a''')^2 + (a'a''b''')^2 + (a'a'''b'')^2 + (a''a'''b')^2 \\ & + (b'b''b''')^2 + (b'b''a''')^2 + (b'b'''a'')^2 + (b''b'''a')^2 \\ & + 2(a'a''b')(b'''a''a''') - 2(a'a''b'')(b'''a'a''') \\ & - 2(a'a'''b')(b''a''a''') + 2(b'b''a')(a'''b''b''') \\ & - 2(b'b''a'')(a'''b'b''') - 2(b'b'''a')(a''b''b'''). \end{aligned}$$

If now we make use of the identity

$$(a'a''b')(a''a'''b''') = (a'a''a''')(a''b'b''') + (a'a''b''')(b'a''a'''),$$

and of the analogous ones

$$\begin{aligned}
(a'a''b'')(b'''a'a''') &= (a'a''a''')(a'b''b''') - (a'a''b''')(a'b''a'''),\\
(a'a'''b')(b''a''a''') &= -\ (a'a''a''')(b'b''a''') - (a'b''a''')(b'a''a'''),\\
(b'b''a')(a'''b''b''') &= -\ (b'b''b''')(a'b''a''') + (b'b''a''')(a'b''b'''),\\
(b'b''a'')(a'''b'b''') &= (b'b''b''')(b'a''a''') - (b'b''a''')(b'a''b'''),\\
(b'b'''a')(a''b''b''') &= (b'b''b''')(a'a''b''') - (b'a''b''')(a'b''b'''),
\end{aligned}$$

the above expansion reduces to the sum of squares

$$[(a'a''a''') - (a'b''b''') + (b'a''b''') - (b'b''a''')]^2$$
$$+ [(b'b''b''') - (b'a''a''') + (a'b''a'') - (a'a''b''')]^2.$$

If we represent by (A) the matrix of the a's, by $(-A)$ that of the $-a$'s, and similarly by (B) and $(-B)$ the matrices of the b's and $-b$'s respectively, the matrix of the determinant D may be represented symbolically as follows:

$$\begin{Vmatrix} (A) & (B) \\ (-B) & (A) \end{Vmatrix}.$$

It still remains to study determinants whose matrices are of the type

$$\begin{Vmatrix} (A) & (B) & (C)\cdots \\ (-B) & (A) & (D)\cdots \\ (-C) & (-D) & (A)\cdots \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \end{Vmatrix}.$$

August, 1914.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY.

Analytic Geometry of Space. By VIRGIL SNYDER and C. H. SISAM. Henry Holt and Co., New York, 1914. xi+289 pages.

It is probable that in no branch of elementary mathematics has there been such need of a good, teachable book as in the analytic geometry of space. Books on this subject, designed for two half-year courses, are strangely lacking. Charles Smith's book on solid geometry gives fine results when used with a small class of picked students because, like so many other text-books from England, it forces the student who would get anything from it to think, to remember and to coördinate many branches of elementary mathematics. In other words, the book is

not made up of a series of mechanical rules and illustrative examples, than which there is probably no influence more stultifying to a student's originality and power to attack a problem intelligently. Moreover, in such a book as Smith's, a student's mental capacity is treated with respect; as his mind becomes more virile, it is not nauseated by continual regurgitation of things once digested.

Smith's book, however, is lacking in many of the qualities and quantities which make the book under review a notable contribution to class-room literature.

Anyone sufficiently interested in analytic geometry to read this review can and probably will find a table of contents in the book itself. Therefore the reviewer will content himself with the privilege of calling attention to some of the exceptional features of the book.

Taken as a whole the book is noteworthy for its literary style. It is a delight to read it. Simple, forceful language is employed throughout, the theorems are models of clear expression and, when a paragraph is completed, its connection with the rest of the subject is apparent.

We are glad to note that, when the proof of some theorem has been standardized on account of its elegance, the authors have not felt obliged to bring forth an outrageous, new demonstration. The reviewer will never agree that desire for change is sufficient excuse to mangle a natural, tried and beautiful development.

The book is arranged so that its contents fall very naturally into materials for two courses, the first eight chapters constituting a course of some thirty-five lessons and the last six one of about fifty lessons.

The outstanding feature of the first eight chapters, and to the reviewer the finest thing in the book, is the natural and immediate introduction of certain geometric concepts which most authors seem to feel are better left out of a first course, such concepts as plane coördinates, homogeneous coördinates of points and planes, elements at infinity, imaginary elements, the absolute, circular points and isotropic planes.

The theorem that the locus of $Ax + By + Cz + D = 0$ (A, B, C, D real) is a plane is proved in a way which is very satisfying and, though a bit tedious, is very natural. The authors show that the locus satisfies three conditions that characterize a plane: (1) It contains three points not on a line; in fact, if $C \neq 0$, the three points $[0, 1, (-B - D)/C]$, $[1, 0, (-A - D)/C]$ and $(0, 0, -D/C)$. (2) It contains every point on any line joining two points on it. (3) It does not contain all points of space; in fact, it does not contain the point $[0, 0, (-D-C/C]$.

In dealing with the normal form of the equation of a plane and the perpendicular distance from a plane to a point, the authors use only the positive $\sqrt{A^2 + B^2 + C^2}$; they introduce the idea of a positive and negative side of the plane by the definition that (x_1, y_1, z_1) is on the positive side of the plane if $Ax_1 + By_1 + Cz_1 + D > 0$.

The point at infinity on a given line is introduced by saying that for every value of k, including $-a_4/b_4$, there is to correspond a point on the line $x_i = (a_i + b_ik)/(a_4 + b_4k)$ $(i = 1, 2, 3)$. Similarly, having introduced homogeneous coördinates as four numbers x', y', z', t' such that $x = x'/t'$, etc., a point is said to be defined even when $t' = 0$, and hence is a point at infinity.

The reviewer wishes especially to commend the early introduction of the so-called imaginary elements. He has always felt that the imagination and hence appreciation of a beginner in any sort of geometry is very greatly stimulated by an early consideration of these elements. It is well that he should be disillusioned at the start of the idea that the word *real* in geometry has any connection with the everyday word *reality*. The authors enforce this concept of imaginary elements by noting that the line determined by two conjugate points or planes is real, and similar theorems. They also furnish a fair list of examples.

The easy swing of the text is illustrated by the paragraph introducing the projecting cylinders of a curve: "The equation of the projecting cylinder of the curve of intersection of two surfaces $F(x, y, z) = 0$, $f(x, y, z) = 0$ on the plane $z = 0$ is independent of z. The equation of this cylinder may be obtained by eliminating z between the equations of the curve."

The absolute is introduced by proving that all spheres (it should be all non-composite spheres, to use the authors' terminology) intersect the plane at infinity in the same curve. The circular points and theorems connected with them, of course, enter very simply at this point. The authors err, I believe, in not making it sufficiently emphatic that all co-planar circles pass through the same circular points, after having defined the circular points as the two points in which any proper circle intersects the absolute.

The radical plane of two spheres is introduced as the ordinary plane of the composite sphere in $\lambda_1 S + \lambda_2 S' = 0$, where $S = 0$ and $S' = 0$ are two spheres, and it is then shown that this plane is the locus of the centers of spheres intersecting the two given spheres orthogonally. If we omit the consideration that this method of approach is general and illustrates the radical plane's relation to a system of spheres, we believe that several other ways of introducing the radical plane are more desirable, because less cumbersome and more directly interesting. The examples under this section are extremely well chosen.

Three chapters are devoted to an elementary discussion of quadrics. These surfaces are introduced by means of their simplest equations and are discussed with reference to their shape. I have yet to find an author who points out in his text how easy it is, by noting the exceptional sign and variable in these equations, to designate the type and position of the quadric. This ability to sketch a surface rapidly is a desideratum in many applications of analytic geometry.

In the treatment of the general Cartesian equation of quadrics, much stress is laid on the distinction between centers and vertices. If $F(x, y, z) = 0$ represents the given quadric, the lines and planes of centers and vertices are introduced by means of the rank of the system of planes $\partial F/\partial x = \partial F/\partial y = \partial F/\partial z = 0$. I might say in passing that this chapter has nearly as many typographical errors as the preceding one has elegant figures and photographs.

The work on the discriminating cubic—one of the brightest spots in the analytic geometry of space—is particularly good, and the discussion of several of the special cases of quadrics is made very properly by means of illustrative examples. Little is gained in such cases by an involved, *general* discussion where difficulties

that do not really exist are apparently brought to light. In the examples connected with this chapter the authors have apparently gone to considerable trouble to get cases whose solutions involve complicated numbers and radicals.

The chapter on properties of quadrics is very concise. The treatment of circular sections deserves mention. It begins with the proof (given by two methods involving different principles) of the theorem: "Through each real, finite point of space pass six planes which intersect a given non-composite, non-spherical quadric in circles. If this quadric is not a surface of revolution nor a parabolic cylinder, these six planes are distinct; two are real and four imaginary. If the quadric is a surface of revolution or a parabolic cylinder, four of the planes are real and coincident and two imaginary." This theorem is then applied to standard forms of the quadric.

The point of view adopted in the ninth chapter and continued throughout the remainder of the book is quite different from that of the first part. Tetrahedral coördinates are here introduced by means of linear transformations; the principle of duality is used and more maturity in the student is demanded. As the authors state in the preface, the selection of subject matter in these last chapters is such as to be of greatest service for further study of algebraic geometry.

The road to an intelligent discussion of this subject matter is paved by two splendidly concise and clear chapters concerning such ideas as tetrahedral coördinates, duality, transformations of coördinates, cross ratio, invariance of the discriminant, lines on a quadric and polar theory. We recommend to anyone who has usually found the introduction of these concepts long and involved a careful perusal of chapters nine and ten of the book under review.

Chapter eleven, on linear systems of quadrics, is divided into four clearly differentiated parts, consisting respectively of pencils, bundles and webs of quadrics, and linear systems of rank r. The discussion of pencils is devoted to their classification into fifteen forms, arranged according to their *characteristic* (the symbol indicating the arrangement of roots in a given λ-discriminant). These forms are eventually tabulated, with a column for the curve of intersection of the two basic quadrics of the system. The culminating theorem of the section on webs of quadrics is that the points of the Kummer and Weddle surfaces are in one to one correspondence.

The reader will find the chapter on transformations one of the most satisfying of the book. It begins with two very illuminating and stimulating pages on the necessary and sufficient conditions that different elements are perpendicular, i. e., on poles and polars as to the absolute. Projective transformations are classified and tabulated in standard forms, fourteen in number, the determination of their characteristics and the loci of their invariant points being left as an exercise to the student. Birational transformations and the geometric constructions for some of them are mentioned very briefly, but in such an attractive manner as to give any student a taste for more.

The discussion of the algebraic surfaces

$$\Sigma \frac{n!}{\alpha!\beta!\gamma!\delta!} a_{\alpha\beta\gamma\delta} x_1^{\alpha} x_2^{\beta} x_3^{\gamma} x_4^{\delta} = 0$$

(α, β, γ, δ integers positive or zero and $\alpha + \beta + \gamma + \delta = n$) and algebraic space curves requires a distinctly more advanced knowledge of plane analytic geometry. When mastered the student will understand what things in general concern an investigator in algebraic curve theory and will have a good working knowledge of the main properties of space cubic and quartic curves.

After such a surfeit of praise as the above contains, the reviewer felt it his duty to look very carefully for something serious to criticize; but what he found seems very trivial. Of course the ubiquitous typographical errors are there and the figure used in finding the distance between two non-intersecting lines would seem to have been drawn by one who did not consult the text of that particular paragraph.

Polar, spherical, and cylindrical coördinates are very nicely introduced at the beginning, but are not mentioned again. If they have a place in the analytic geometry of space, that place should be at least visible.

Practically all the problems have the answers given and there are not enough of them incorrect to be of any pedagogical value. The reviewer would like to see the pages of this MONTHLY opened to a discussion of the relative pedagogical values of complete, partial and no lists of answers, and of answers all correct or partially incorrect.

The definition of analytic curves in the last short chapter on differential geometry is obviously made so as not to alarm the reader and is not precise. Incidentally it is made without any explanation of what an analytic function is.

For American students, at least, I feel very strongly that such a book would gain much in effectiveness if the first chapters were devoted to a concise discussion of those theorems on determinants and matrices that will be of service in the succeeding chapters. A teacher, having in mind what is to follow, has a great opportunity to make his students appreciate the great power and elegance of determinants.

Let me say in conclusion that, with its splendid style, its fine choice and arrangement of material and its pedagogical excellences, I believe this book one of the best contributions to American text-books made in recent years.

E. GORDON BILL.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

PROBLEMS FOR SOLUTION.

Note.—Of all the problems proposed between January 1, 1913, and January 1, 1915, satisfactory solutions for the following have not been received:

In Algebra, 406.

In Geometry, 427, 442, 446, and 454.

In Calculus, 339, 340, 342, 348, 353, 360, 363, and 368 to 375.

In Mechanics, 277–8–9, 287, and 291 to 301.

In Number Theory, 191–2, 196, 198, 205, 208–9–10–11, 214, 217, and 219 to 225.

Please give attention to these as well as to those proposed since January 1, 1915.

ALGEBRA.

433. Proposed by B. J. BROWN, student at Drury College.

Prove that, if all the quantities, a, b, etc., are real, then all the roots of the equations

$$\begin{vmatrix} a-x & h \\ h & b-x \end{vmatrix} = 0, \qquad \begin{vmatrix} a-x & h & g \\ h & b-x & f \\ g & f & c-x \end{vmatrix} = 0$$

are real; and generalize the proposition.

434. Proposed by S. A. JOFFE, New York City.

Express the "difference of zero," $\Delta^n 0^{n+1}$, in the form, $C_1(n+2)! - C_2(n+1)!$, where C_1 and C_2 are numerical coefficients independent of n.

435. Proposed by C. N. SCHMALL, New York City.

Show that $(e-1) - \frac{1}{2}(e-1)^2 + \frac{1}{3}(e-1)^3 - \cdots = 1$, where e is the Naperian base of logarithms.

GEOMETRY.

463. Proposed by B. J. BROWN, student at Drury College.

If μ and ν are the parameters of the two confocal conics through any point on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

show that $\mu + \nu + a^2 + c^2 = 0$, along a central circular section.

464. Proposed by FRANK R. MORRIS, Glendale, Calif.

The sum of the hypotenuse and one side of a right triangle is 100 feet. A point on the hypotenuse is 10 feet from each of the sides. Find the length of the hypotenuse correct to the third decimal place.

465. Proposed by ROGER A. JOHNSON, Western Reserve University.

Let C be a fixed circle, A a point outside it. Let AT and AT' be the tangents from A to the circle, touching the latter at T and T'. Let two secants be drawn through A, cutting the circle at P, Q and R, S respectively. Let PR and QS meet at X, PS and QR meet at Y. Prove by elementary methods that for all positions of the secants, X and Y lie on the line TT'.

CALCULUS.

383. Proposed by WILLIAM CULLUM, Albion, Mich.

Find the area of the curved surface of a right cone whose base is the asteroid, $x^{2/3} + y^{2/3} = a^{2/3}$, and whose altitude is h.

From Townsend and Goodenough's *First Course in Calculus*, p. 288, Ex. 11.

Note.—Among other methods, find the required area by means of the formula

$$\int \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy.$$

EDITORS.

384. Proposed by JOSEPH B. REYNOLDS, Lehigh University.

In what time will a sum of money double itself at 6 per cent. interest per annum if compounded at indefinitely short intervals?

385. Proposed by H. B. PHILLIPS, Massachusetts Institute of Technology.

If $f(x)$ is continuous between a and x, show that

$$\operatorname*{Lim}_{n \doteq \infty} \frac{n!}{(x-a)^n} \int_a^x \cdots \int_a^x f(x) dx^n = f(a).$$

386. Proposed by HERBERT N. CARLETON, Newberry, Mass.

C is a fixed point on the perpendicular bisector of the line segment AB. Locate a point D also on this bisector, such that $AD + BD + DC$ shall be a minimum.

MECHANICS.

309. Proposed by JOS. B. REYNOLDS, Lehigh University.

The tangent at one cusp of a vertical, three arched hypocycloid is horizontal, and a particle will just slide under gravity from the upper cusp to this cusp. Find the equation which the coefficient of friction must satisfy.

310. Proposed by EMMA M. GIBSON, Drury College.

A particle movable on a smooth spherical surface of radius a is projected along a horizontal great circle with a velocity v which is great compared with $\sqrt{2ga}$. Prove that its path lies between this great circle and a parallel circle whose plane is approximately at a depth $2ga^2/v^2$ below the center.

From Lamb's *Dynamics*, p. 334, Ex. 3.

SOLUTIONS OF PROBLEMS.

Note.—When several persons send in solutions for the same problem, the committee naturally and, we think, properly select for publication that one which is not only correct mathematically but is written out in the best form for publication. They must either do this or else, if they select solutions which are in poor form, in order to give as many solvers as possible a chance, they must write these solutions over to save them from rejection by the Managing Editor as bad copy. The task of *putting solutions into acceptable shape for the printer* is one which the members of the committee do not relish,—and who can blame them? This will explain why some names appear more frequently at the head of the solutions than do others, even though several may have solved the same problem. See the suggestions for preparing solutions published in several previous issues. MANAGING EDITOR.

ALGEBRA.

409. Proposed by C. E. GITHENS, Wheeling, W. Va.

Find integral values for the edges of a rectangular parallelopiped so that its diagnoal shall be rational.

REMARKS BY W. C. EELLS, U. S. Naval Academy.

In the February, 1915, issue of the MONTHLY (pp. 60–61) Artemas Martin criticizes my solution of this problem in the October, 1914, issue, stating that I have solved a different problem from the one proposed, and that I claimed that a certain rational parallelopiped was the smallest possible one, whereas he exhibits four others that are smaller.

Since the problem reads "Find integral values" and not "Find *all* integral values, etc." it was in order to impose the condition $x^2 + y^2 = k^2$ or any other condition so long as integral solutions of the equation $x^2 + y^2 + z^2 = a^2$ were found. I showed two general methods of solution, each giving an infinite number of *prime* integral solutions, but did not state nor even suppose that I had found all possible solutions. I fail to see, however, how I solved a *different* problem from the one proposed.

Under my first method, as an example, I gave the solution $(x, y, z, a) = (4, 3, 12, 13)$ as the smallest rational parallelopiped, and it should have been sufficiently

evident that what I meant was that this was the smallest one found by this method,—not the smallest one possible. The latter would have been a rash and unwarranted statement, when I was not professing to give *all* solutions. Mr. Martin cites the four solutions (1, 2, 2, 3), (2, 3, 6, 7), (1, 4, 8, 9), (2, 6, 9, 11) to prove that (3, 4, 12, 13) is not the smallest. He might have cited two more, (4, 4, 7, 9) and (6, 6, 7, 11).

Mr. Martin gives the equation

$$a^2 + b^2 + \left(\frac{a^2 + b^2 - c^2}{2c}\right)^2 = \left(\frac{a^2 + b^2 + c^2}{2c}\right)^2,$$

"true for all values of a, b, c," from which to derive solutions of the equation $x^2 + y^2 + z^2 = d^2$. This identity is true but quite useless unless further and quite elaborately qualified. For many values of a, b, c, it gives z and d fractional values in violation of the conditions of the problem. Mr. Martin gives no restrictions on values of a, b, c, the obvious implication being that one is free to assign values to them at pleasure. For $a = 5$, $b = 3$, $c = 1$, it gives $z = 33/2$, $d = 35/2$. As a matter of fact there are *no* integral solutions possible for a and b both odd. For let $a = 2m + 1$, $b = 2n + 1$, and let c be even, $c = 2p$. Then

$$z \equiv \frac{4m^2 + 4n^2 + 4m + 4n + 2 - 4p}{4p} \equiv \frac{4K + 1}{2p},$$

evidently impossible as an integral solution. Again let c be odd, $c = 2p + 1$. As before

$$z = \frac{4K + 1}{2(2p + 1)},$$

also impossible. Thus there are no solutions for a and b both odd. Similarly it can easily be shown that there are *no* solutions for c even, when a and b are even-odd or odd-even, but only when a and b are even-even, and not always then. When c is odd there are no solutions if a and b are both even, but only when a and b are even-odd or odd-even. It requires further careful discrimination to properly restrict the form of a and b in the two cases indicated as sometimes yielding solutions.

Soon after submitting the solution to which Mr. Martin objects I set to work to devise a method for finding *all* solutions, worked one out along the lines suggested above, and calculated all possible rational prime solutions, 74 in number for which the diagonal is less than 50. (My previous methods had given only three such.) But on submitting this to the MONTHLY some two months ago Professor R. D. Carmichael kindly called my attention to an elegant solution of the same problem by V. A. Le Besque which was published in the *Comptes Rendus* in 1868 (Vol. 66, pp. 396–398). Le Besque's method was so superior to the one I had devised (as well as to the one proposed by Mr. Martin) that I did not think mine worth publishing and did nothing further with it.

Since the problem has come up again, however, it may be of interest and

value to publish in the MONTHLY Le Besque's method. He gives the following identity,

$$(\alpha^2+\beta^2+\gamma^2+\delta^2)^2 \equiv (\alpha^2+\beta^2-\gamma^2-\delta^2)^2+4(\alpha\gamma+\beta\delta)^2+4(\alpha\delta-\beta\gamma)^2.$$

Since every integral number n can be expressed in the form

$$n=\alpha^2+\beta^2+\gamma^2+\delta^2, \quad (\alpha, \beta, \gamma, \delta = 0, 1, 2, 3, \cdots)$$

this affords an easy and rapid means of finding integral sides for any integral diagonal n. Le Besque's identity needs certain restrictions which need not be stated here, on the form and relative size of α, β, γ, δ, to avoid duplication and results not relatively prime. If we put $\delta = 0$ it affords a still more rapid method of finding an indefinite number of solutions, although of course not all of them. For $\beta=\delta=0$ it gives the well-known right triangle solutions, $\alpha^2+\gamma^2$, $\alpha^2-\gamma^2$, $4\alpha\gamma$.

FURTHER REMARKS BY HERBERT N. CARLETON, Newberry, Mass.

Mr. Martin's formula can be much simplified and brought to a form in which two numbers representing two of its sides can be directly derived.

Thus, let a, b, c be the three edges. Then since $(a+b)^2 \equiv a^2+2ab+b^2$, it is only necessary to choose a and b so that $2ab = \square$. When such values of a and b are determined, the third edge, $c=\sqrt{2ab}$, and we have

$$a^2+b^2+c^2=\square.$$

Since $2ab = a^2\cdot\dfrac{2b}{a}$, if $a^2=\dfrac{2b}{a}$, or $b=\frac{1}{2}a^3$, an integer, the conditions are fulfilled. From this it is seen that a^3 must be even and therefore a must be even.

Hence, by taking a equal to any even number, $b=\frac{1}{2}a^3$, and $c=\sqrt{2ab}$, we get numbers satisfying the conditions of the problem.

Note.—Each of these methods of solution has value and each satisfies the conditions of the problem, and none of them, it appears, will by any direct method include all possible solutions. Such a solution, so far as we know, does not exist. EDITORS.

420. Proposed by ELBERT H. CLARKE, Purdue University.

Given the infinite series,

$$\frac{a}{r}+\frac{b}{r^2}+\frac{a+b}{r^3}+\frac{a+2b}{r^4}+\frac{2a+3b}{r^5}+\cdots,$$

in which a and b are any numbers and where each numerator after the second is the sum of the two preceding numerators. To find the region of convergence and the sum of the series.

This problem is a generalization of one solved in the January (1914) number of the MONTHLY.

SOLUTION BY MRS. ELIZABETH B. DAVIS, U. S. Naval Observatory.

For $r \leqq 1$, the series is divergent. For $r>1$, the series is convergent. Let

$$S=\frac{a}{r}+\frac{b}{r^2}+\frac{a+b}{r^3}+\frac{a+2b}{r^4}+\cdots.$$

We have

$$\frac{a+b}{r^3} - \left(\frac{b}{r^2}\right)\cdot\frac{1}{r} - \left(\frac{a}{r}\right)\cdot\frac{1}{r^2} = 0,$$

$$\frac{a+2b}{r^4} - \left(\frac{a+b}{r^3}\right)\cdot\frac{1}{r} - \left(\frac{b}{r^2}\right)\cdot\frac{1}{r^2} = 0,$$

$$\cdots\cdots\cdots\cdots\cdots$$

Therefore, the scale of relation is $1 - (1/r) - (1/r^2)$. Multiplying S successively by each term of the scale of relation, we have

$$S = \frac{a}{r} + \frac{b}{r^2} + \frac{a+b}{r^3} + \frac{a+2b}{r^4} + \cdots,$$

$$-\frac{1}{r}\cdot S = -\frac{a}{r^2} - \frac{b}{r^3} - \frac{a+b}{r^4} - \cdots,$$

$$-\frac{1}{r^2}\cdot S = -\frac{a}{r^3} - \frac{b}{r^4} - \cdots.$$

Adding, we have

$$\left(1 - \frac{1}{r} - \frac{1}{r^2}\right)\cdot S = \frac{a}{r} + \frac{b-a}{r^2}.$$

Multiplying by r^2, we have

$$(r^2 - r - 1)\cdot S = a(r-1) + b.$$

Hence,

$$S = \frac{a(r-1)+b}{r(r-1)-1}.$$

Applying this formula to the special series solved in the January (1914) number of the MONTHLY, in that example

$$a = 1, \qquad b = 2, \qquad r = 2,$$

and the scale of relation $= 1 - (1/2) - (1/4)$; whence

$$S = 1 + \frac{1(2-1)+2}{2(2-1)-1} = 1 + 3 = 4.$$

GEOMETRY.

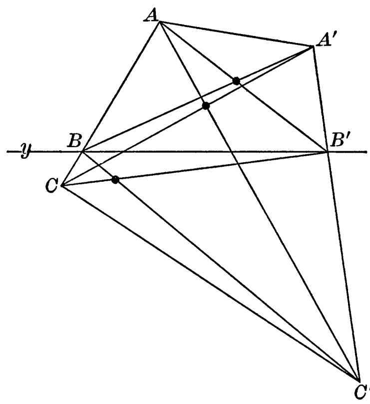
443. Proposed by CHARLES N. SCHMALL, New York City.

A quadrilateral of any shape whatever is divided by a transversal into two quadrilaterals. The diagonals of the original figure and those of the two resulting (smaller) figures are then drawn. Show that their three points of intersection are collinear.

II. SOLUTION BY VOLA BARTON, Goucher College.

Quadrilateral $AA'C'C$ is cut by line y. AC and $A'C'$ are cut by y in B and B', respectively. A, B, and C are collinear, and also A', B', and C', by hypothesis. These six points are so joined as to form the hexagon $AB'CA'BC'$.

The intersections of the opposite sides are collinear, by the theorem of Pappus, that is, if a hexagon $AB'CA'BC'$ has its vertices of odd order on one straight line,



and its vertices of even order on another straight line, then the three pairs of opposite sides, AB' and $A'B$, $B'C$ and BC', CA' and $C'A$, meet in three points lying on another straight line.

But the intersections of the opposite sides of this hexagon are the intersections of the diagonals of the three quadrilaterals.

Hence, the intersections of the diagonals of any three quadrilaterals, two of which are formed by cutting the other one by a straight line, are collinear.

Also solved by ANNA MULLIKIN.

A solution of this problem appeared in the January issue of the MONTHLY, but we publish this solution as it presents an entirely different method of attack. EDITORS.

452. Proposed by NATHAN ALTSHILLER, University of Washington.

Through a given point a secant is drawn that meets three given concurrent lines in the points A, B, C, respectively. Determine the position of the secant by the condition $AB/BC = K$, K being given.

SOLUTION BY MRS. ELIZABETH B. DAVIS, U. S. Naval Observatory.

Let OA', OB' and OC' be three given concurrent lines, and P a given point. Let it be required to draw through P a secant meeting OA', OB' and OC' respectively in points A, B, and C, such that $AB/BC = K$, K being given.

Join P and O, and through P draw any transversal $R'P$, meeting the four lines of the pencil $O - A'B'C'P$ in D, E, F, and P, respectively.

On OP take H and G so that

$$GP : HP = DE : EF. \qquad (1)$$

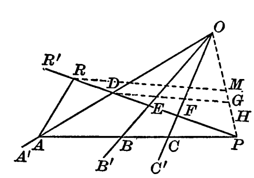
Also, on OP take M, so that

$$MP : HP = K. \qquad (2)$$

Join DG, and through M draw RM parallel to DG, meeting $R'P$ in R.

Draw RA parallel to OC', meeting OA' in A. Join AP, then AP is the transversal required. For, dividing (1) by (2), we have

$$GP : MP = \frac{DE}{EF} : K.$$



Since, $\triangle$'s PDG and PRM are similar,

$$GP : MP = DP : RP.$$

Hence

$$DP : RP = \frac{DE}{EF} : K. \qquad (3)$$

Dividing the first ratio of (3) by FP,

$$\frac{DP}{FP} : \frac{RP}{FP} = \frac{DE}{EF} : K. \qquad (4)$$

Since $\triangle$'s RAP and FCP are similar,

$$\frac{RP}{FP} = \frac{AP}{CP}.$$

Substituting this value of RP/FP in (4), we have

$$\frac{DP}{FP} : \frac{AP}{CP} = \frac{DE}{EF} : K,$$

or

$$\frac{DP}{FP} : \frac{DF}{EF} = \frac{AP}{CP} : K. \qquad (5)$$

But the anharmonic ratio of all transversals cutting any given pencil of four lines, as $O-A'B'C'P$, is a constant. Hence,

$$\frac{DP}{FP} : \frac{DE}{EF} = \frac{AP}{CP} : \frac{AB}{BC}. \qquad (6)$$

From (5) and (6), $AB/BC = K$. Hence AP is the transversal required.

Also solved by the Proposer.

453. Proposed by CLIFFORD N. MILLS, Brookings, S. D.

Prove geometrically the formulæ for $\sin 2\beta$, $\cos 2\beta$, $\sin 3\beta$, $\cos 3\beta$.

Solution by the Proposer.

(1) $\sin 2\beta = 2 \sin \beta \cos \beta$.

Inscribe in a circle any triangle, ABC, with the angle at B equal to 2β (Fig. 1). Draw AM through the center of the circle; BK, the bisector of the angle B; OK, the radius of the circle; AK, KM, and KC, K being on the circumference of the circle.

Then

$$\sin 2\beta = \frac{AC}{2r}.$$

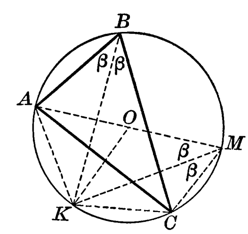


Fig. 1.

△ AKC and KOM are similar isosceles triangles. Hence

$$\frac{AC}{AK} = \frac{KM}{r} \quad \text{or} \quad AC = \frac{AK \cdot KM}{r}.$$

Hence,

$$\frac{AC}{2r} = \frac{AK \cdot KM}{2r^2} = \frac{2AK \cdot KM}{4r^2}.$$

But

$$\frac{AK}{2r} = \sin \beta \quad \text{and} \quad \frac{KM}{2r} = \cos \beta.$$

Hence $\sin 2\beta = 2 \sin \beta \cos \beta$.

(2) $\cos 2\beta = \cos^2 \beta - \sin^2 \beta$.

Using the same Fig. 1, we get

$$\cos 2\beta = \frac{MC}{2r} = \sqrt{\frac{4r^2 - \overline{AC}^2}{2r}} = \sqrt{1 - \frac{\overline{AC}^2}{4r^2}}.$$

But

$$AC = (AK \cdot KM)/r.$$

Hence,

$$\cos 2\beta = \sqrt{1 - \frac{4\overline{AK}^2 \cdot \overline{KM}^2}{16r^4}}.$$

Substituting the values of AK and KM, as previously found, we have

$$\cos 2\beta = \sqrt{1 - 4 \sin^2 \beta \cos^2 \beta} = \sqrt{(\sin^2 \beta + \cos^2 \beta)^2 - 4 \sin^2 \beta \cos^2 \beta} = \cos^2 \beta - \sin^2 \beta.$$

(3) $\sin 3\beta = 3 \sin \beta - 4 \sin^3 \beta$.

Inscribe in a circle any triangle ABC with the angle at B equal to 3β (Fig. 2). Draw AM, the diameter of the circle. Arc AH = arc HK = arc KC. Draw the chords AK, HC, KC, HM, KM, BH, and BK.

Hence,

$$\sin 3\beta = \frac{AC}{2r}.$$

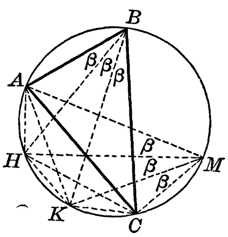


FIG. 2.

$AHKC$ is a concylic quadrilateral. Hence,

$$AC\cdot HK + AH\cdot KC = AK\cdot HC,$$

or

$$AC\cdot AH + \overline{AH}^2 = AK\cdot HC = \overline{AK}^2;$$

whence,

$$AC = (\overline{AK}^2 - \overline{AH}^2)/AH.$$

Hence,

$$\sin 3\beta = (\overline{AK}^2 - \overline{AH}^2)/2r\cdot AH.$$

But

$$AK = HC = 2r \sin 2\beta = 4r \sin\beta\cos\beta,$$

and

$$AH = 2r\sin\beta.$$

Hence,

$$\sin 3\beta = \frac{16r^2\sin^2\beta\cos^2\beta - 4r^2\sin^2\beta}{4r^2\sin\beta} = 3\sin\beta - 4\sin^3\beta.$$

(4) $\cos 3\beta = 4\cos^3\beta - 3\cos\beta$.

From figure 2, we have

$$\cos 3\beta = \frac{MC}{2r}.$$

Also, since $MC^2 = AM^2 - AC^2$,

$$\frac{\overline{MC}^2}{4r^2} = \frac{4r^2 - \overline{AC}^2}{4r^2}.$$

But

$$\overline{AC}^2 = \frac{\overline{AK}^4 - 2\overline{AK}^2\cdot\overline{AH}^2 + \overline{AH}^4}{\overline{AH}^2}.$$

Hence,

$$\frac{\overline{MC}^2}{4r^2} = 1 - \frac{\overline{AK}^4}{4r^2\overline{AH}^2} + 2\frac{\overline{AK}^2}{4r^2} - \frac{\overline{AH}^2}{4r^2}.$$

Substituting the values of AK and AH, we have

$$\frac{\overline{MC}^2}{4r^2} = 1 - 16\sin^2\beta\cos^4\beta + 8\sin^2\beta\cos^2\beta - \sin^2\beta = 16\cos^6\beta - 24\cos^4\beta + 9\cos^2\beta,$$

or

$$\frac{MC}{2r} = 4\cos^3\beta - 3\cos\beta.$$

Hence,

$$\cos 3\beta = 4\cos^3\beta - 3\cos\beta.$$

Also solved by A. M. HARDING and J. VINCENT BALCH.

CALCULUS.

366. Proposed by I. A. BARNETT, University of Chicago.

Compute the definite integral $\int_a^b \sin^{-1} x dx$, where $0 \leqq a \leqq 1$, $0 \leqq b \leqq 1$, by direct summation.

SOLUTION BY THE PROPOSER.

Let $\sin A = a$, or $A = \sin^{-1} a$, and let $b = \sin (A + nh) = \sin B$. Hence $n = (B - A)/h$.

Then we have

$$\begin{array}{cccc} a & & & b \\ \sin A & \sin (A + h) \cdots & \sin [A + (n - 1)h] & \sin (A + nh) \end{array}$$

$$\begin{aligned}
\int_a^b \sin^{-1} x dx &= \lim_{h \doteq 0} [\{\sin (A + h) - \sin A\} \sin^{-1} \sin A \\
&\quad + \{\sin (A + 2h) - \sin (A + h)\} \sin^{-1} \sin (A + h) + \cdots \\
&\quad + \{\sin (A + nh) - \sin [A + (n - 1)h]\} \sin^{-1} \sin [A + (n - 1)h]] \\
&= \lim_{h \doteq 0} [- h\{\sin (A + h) + \sin (A + 2h) + \cdots + \sin (A + nh)\} - A \sin A \\
&\quad + (A + nh) \sin (A + nh)] \\
&= \lim_{h \doteq 0} \left[- \frac{h/2}{\sin h/2} \left\{ \cos \left(A + \frac{h}{2} \right) - \cos \left[A + (2n + 1) \frac{h}{2} \right] \right\} - A \sin A \right. \\
&\quad \left. + (A + nh) \sin (A + nh) \right] \\
&= \lim_{h \doteq 0} \left[- \frac{h/2}{\sin h/2} \left\{ \cos \left(A + \frac{h}{2} \right) - \cos \left(B + \frac{h}{2} \right) \right\} - A \sin A + B \sin B \right] \\
&= \cos B - \cos A + B \sin B - A \sin A \\
&= \sqrt{1 - b^2} - \sqrt{1 - a^2} + b \sin^{-1} b - a \sin^{-1} a,
\end{aligned}$$

which agrees with the result given in the tables.

An excellent solution was also received from A. M. HARDING.

367. Proposed by C. N. SCHMALL, New York City.

Show that the volume inclosed by the surface $(x^2 + y^2 + z^2)^5 = (a^3x^2 + b^3y^2 + c^3z^2)^2$ is $\frac{4}{9}\pi(a^3 + b^3 + c^3)$.

SOLUTION BY A. M. HARDING, University of Arkansas, AND A. R. NAUER, St. Louis, Mo.

Let $x = r \sin \varphi \cos \theta$, $y = r \sin \varphi \sin \theta$, $z = r \cos \varphi$. Substituting in the given equation, we obtain

$$r^3 = a^3 \sin^2 \varphi \cos^2 \theta + b^3 \sin^2 \varphi \sin^2 \theta + c^3 \cos^2 \varphi.$$

Then the volume inclosed by the surface is given by

$$\begin{aligned}
V &= 4 \int_0^{\pi/2} \int_0^{\pi} \int_0^r r^2 \sin \varphi dr d\theta d\varphi, \\
&= \frac{4}{3} \int_0^{\pi/2} \int_0^{\pi} (a^3 \sin^2 \varphi \cos^2 \theta + b^3 \sin^2 \varphi \sin^2 \theta + c^3 \cos^2 \varphi) \sin \varphi d\theta d\varphi, \\
&= \frac{4}{3} \int_0^{\pi/2} \left(\frac{\pi}{2} \cdot a^3 \sin^3 \varphi + \frac{\pi}{2} b^3 \sin^3 \varphi + \pi c^3 \cos^2 \varphi \sin \varphi \right) d\varphi, \\
&= \frac{4\pi}{3} \left[\frac{a^3}{2} \cdot \frac{2}{3} + \frac{b^3}{2} \cdot \frac{2}{3} + \frac{c^3}{3} \right] = \frac{4\pi}{9} (a^3 + b^3 + c^3).
\end{aligned}$$

NUMBER THEORY.

218. Proposed by ELIJAH SWIFT, University of Vermont.

If p is a prime >3, show that $\sum_{a=1}^{p-1}\frac{1}{a^2}\equiv 0 \pmod{p}$.

II. Solution by the Proposer.

It may be proved (Bachmann, *Niedere Zahlentheorie*, page 155) that $A_{p-3}=1\cdot 2\cdot 3\cdots p-3+1\cdot 3\cdot 4\cdots p-2+\cdots$, taking all possible combinations of the numbers 1, 2, $\cdots p-1$, $p-3$ at a time, is divisible by p. Then

$$A_{p-3}=(p-1)!\sum_{\substack{a=1\\ b=1}}^{\substack{a=p-1\\ b=p-1}}\frac{1}{a\cdot b}\qquad a\neq b.$$

Then

$$\sum\frac{1}{a\cdot b}\equiv 0 \pmod{p}.$$

This sum may be written

$$\frac{1}{2}\left\{\sum_{b=1}^{p-1}\frac{1}{1\cdot b}-\frac{1}{1^2}+\sum_{b=1}^{p-1}\frac{1}{2\cdot b}-\frac{1}{2^2}+\cdots+\sum_{b=1}^{p-1}\frac{1}{(p-1)b}-\frac{1}{(p-1)^2}\right\}.$$

But

$$\sum_{b=1}^{p-1}\frac{1}{a\cdot b}\equiv\sum_{b=1}^{p-1}\frac{b}{a}\equiv\sum_{b=1}^{p-1}b\equiv 0 \pmod{p}.$$

Hence,

$$\sum\frac{1}{ab}\equiv-\sum_{a=1}^{p-1}\frac{1}{a^2}\equiv 0 \pmod{p}.$$

QUESTIONS AND DISCUSSIONS.

Edited by U. G. Mitchell, University of Kansas.

NEW QUESTIONS.

26. Why should not the nomenclature of mathematics be made uniform? For example, why call a circle a *portion of a plane* in elementary geometry and a *curve* in analytic geometry? Why call a sphere a *ball* at one time and a *surface* at another time? And so on through all the configurations of two- and three-dimensional geometry.

REPLIES.

18. In view of the present pressure for saving time and gaining efficiency, what are the most important sources of economy in the mathematical courses of the high school and the first two years in college?

Reply by Ernest W. Ponzer, Leland Stanford Junior University.

This question assumes the existence and application of the fundamental principles of efficiency[1] in connection with our college instruction in mathematics. No doubt the editors are aware of the fact that many college professors would hardly make this assumption and that many others would regard the problem involved as quite trivial. If such is not the case, why, for instance, in many large universities is instruction in the mathematics of the freshman year given to a majority of the students enrolled in these courses by assistants in the depart-

[1] Cf. article "A Study in Efficiency," *School Science and Mathematics*, October, 1910, pp. 579–81.

ment? And why is the ferment for a square deal to those courses now working so strongly among those vitally interested, the students themselves and the people who support the institutions? We well know that any assumption of high efficiency in this work is an assumption contrary to fact and that we shall be held responsible for our methods and our results.

Shall we assume the problem to exist, to be worth while, and not to be regarded as a chore in an otherwise interesting round of labors? We must and do; in fact, we believe it to be a fundamental problem in the building of a solid foundation for the mathematical education of the student.

To obtain efficiency we suggest starting with the instructor. We do not believe that the instruction of the first two years should be given by assistants with little or no experience in teaching the subjects. It is worthy of the best efforts of the ablest and most mature instructors in the department. We suggest a recognition of the value of this work and we would accord credit (and salary) to show our appreciation when it is efficiently done. The instructor's attitude toward his work should first of all be sympathetic and he should have the courage of his convictions in carrying it out.

The instructor should have good raw material on which to work. While he must take all who come with the proper credentials, he can eliminate the weaker students. At Stanford for a number of years we have eliminated each fall from our first year sections in applied mathematics on an average about ten per cent. of the students registered. This is done after a review of secondary algebra extending over about three weeks concluded by an examination. The ten per cent. were certainly not at that time up to the standard required for efficient work. Those eliminated had to get a better preparation. Their tales of why they could not at that time hope to handle their mathematics as they should are interesting tales of inefficient work done in high schools. You have all heard these. The stimulus of the sifting process on the ninety per cent. remaining is great. They have drawn a bonus and are alert for more. Their attitude toward their work is excellent. The instructor can begin to see the gradual working of the refining process which is to extend for a year through algebra, review of trigonometry, and analytic geometry, with perhaps the elements of the calculus. Sections now average about 30—small enough for individual and laboratory work and not large enough to require a lecture treatment. Better have fewer good instructors and more good student assistants than many teaching assistants.

We would have student assistants to read papers and exercises handed in. Necessarily these are handed in on time—likewise returned on time. A schedule of assignments based on the text in use is early placed in the hands of students. And they are held responsible for their share in an intelligent discussion of the assignments from day to day.

Perhaps the methods of the instructor in conducting the courses should vary with the class of students enrolled. My experience has been almost exclusively with classes whose major interest was in applied science, especially those electing engineering as their major. An instructor should acquaint himself with the aims

of his particular classes of students and then adapt his methods to suit their needs. For instance, and for applied mathematics: "The problems of applied science call for quantitative results. Abstract and formal results are not sufficient. They must be correct to the proper decimal point. They should check. Accuracy and efficiency must obtain. Form and logical sequence must be seriously considered. Suitable methods whether graphical, numerical, or mechanical are necessary. The province of applied mathematics is to furnish such suitable methods. And the field is not at all to be considered as apart from pure mathematics but a part of it. It is simply a question of a different state of mind handling its problems efficiently"—I quote (translated) from Runge, of Göttingen, Germany.

A teaching assistant whose main interest is centered on the courses in advanced work in mathematics which he himself is taking will too often fail to hit the nail on the head. Efficiency in instruction will too often be low. And no one is deceived by the procedure—least of all the freshman student who is laying a foundation for his future work.

Early in the work of the first year; in fact during that of the first semester, I have made it a practice to insist on the checking of results by the aid of the slide rule. At the end of the year all handle this phase of the work efficiently. This checking is fundamental, that is the way the engineer works; why not use this tool early? Elaborate checks, such as are frequently given in the solution of triangles in trigonometry, are quite unnecessary to the efficient student who applies his common sense and his slide rule as he goes. A similar word might be said for the planimeter which naturally is used more in connection with the calculus, not to mention many other mechanical aids and tables. These are not to be used for first solutions but as efficient aids in checking. Since the work is thus done where mathematics is applied, why should we fall so far short in our instruction? A fair trial will convince the most skeptical or indifferent instructor of the value of all such aids.

I have confined my remarks mainly to the work of a first year course, for I believe that the instructor can there either make or mar, to a large extent, the work of the student in the later years of his course. Certainly he can greatly influence its efficiency. And the student certainly has a right to expect that he will get a square deal from the beginning instead of having the courses he takes treated with indifference by those in charge of them.

I would call for high grade and efficient instruction all along the line. And the best feature of it all is that the students will appreciate the standards set and the efforts of all of us toward securing this higher efficiency.

3. In connection with the theory of the conduction of electricity through gases, one is led to the differential equation

$$y\frac{d^2y}{dx^2} + a\left(\frac{dy}{dx}\right)^2 + b\frac{dy}{dx} + cy + d = 0, \tag{1}$$

where a, b, c, d are constants. For unrestricted values of a, b, c, d the solution of this differential equation presents peculiar difficulties, the series solutions obtained by the customary methods

having (apparently) too small a range of convergence to be satisfactory from the point of view of electrical theory. The general solution of this equation is wanted in case it can be found. If no general solution is obtained for unrestricted a, b, c, d, it is desirable to know special values of a, b, c, d or special relations among a, b, c, d which make it possible to find the general solution; and this solution is desired in each case.

Reply by W. W. Beman, University of Michigan.

The given equation may be written

$$yy'' + ay'^2 + by' + cy + d = 0.$$

Put $y' = p$. Whence $y'' = p\dfrac{dp}{dy}$.

Then

$$yp\frac{dp}{dy} + ap^2 + bp + cy + d = 0$$

and

$$\frac{pdp}{cp^2 + bp + cy + d} + \frac{dy}{y} = 0.$$

If c be 0, the variables are separated. If now $p = f(y)$ or $y = \phi(p)$, we may be able to integrate again, expressing y as a function of x, or x and y as functions of p.

It is interesting to note that the integration called for here in terms of y and p is the same as the one involved in the solution given in the February number of the Monthly (page 72) when $a = 1$.

Also, Professor C. E. Love of the University of Michigan makes the following contribution toward the solution of this equation.

Writing the equation, for greater distinctness,

$$yy'' + \alpha y'^2 + \beta y' + \gamma y + \delta = 0$$

and taking y as the independent variable,

$$yx'' - \alpha x' - \beta x'^2 - (\gamma y + \delta)x'^3 = 0.$$

If $\beta = 0$, we have the linear extended form. Whence

$$\frac{1}{x'^2} = \frac{-2\gamma y}{2\alpha + 1} - \frac{\delta}{\alpha} + Cy^{-2\alpha},$$

$$x = \int \frac{dy}{\sqrt{-\dfrac{2\gamma y}{2\alpha + 1} - \dfrac{\delta}{\alpha} + Cy^{-2\alpha}}}$$

which can be evaluated if $\alpha = -1$ or $\alpha = -\frac{1}{4}$.

DISCUSSIONS.

RELATING TO THE TEACHING OF AXONOMETRY.

By Virgil Snyder, Cornell University.

In the January number of the Monthly (page 36) I notice an inquiry whether axonometry is taught anywhere else than at Washington University.

About ten years ago one of our trustees, Mr. W. O. Kerr, suggested that the department of mathematics introduce a regular course in descriptive geometry as an alternative for those Arts students who ought to have some mathematics, but who found the calculus too difficult or too unattractive to be studied with profit. Before the arrangements were completed, Mr. Kerr died, so that we were deprived of his assistance; but the department gave me the task of working out the course and it has been given every second year since then.

This work is not to overlap with the instruction given in descriptive geometry by the technical colleges of the university, but is planned to furnish an insight into the processes and methods of graphical representation of various kinds. As now given the course comprises four chapters: orthogonal projection, plane projection, perspective, and axonometry. The last chapter was given in fifteen lessons and six drawing periods. We took up orthogonal and oblique representation and did considerable reading on military and cavalier projections.

Pedagogically the experiment has been interesting. A considerable number of students took the course who were not taking other work in mathematics and a goodly number of them are now taking further courses. My smallest class had six members, and the largest, twenty-eight.

RELATING TO THE DEFINITION OF A REGULAR CONVEX POLYHEDRON.

By Harrison E. Webb, Central High School, Newark, N. J.

The definition of a regular convex polyhedron is usually given as "a convex polyhedron whose faces are congruent regular polygons, *and whose polyhedral angles are equal.*" It appears that the italicized part of this definition is redundant.

The character of a polyhedral angle as a *configuration* rather than as a *magnitude* is rarely made clear. There doubtless is such a thing as "solid angularity," as it could be measured in terms of the area of the spherical polygon intercepted by the faces of a polyhedral angle at the center of the sphere. But this notion is of no aid to the above definition. What is usually meant is that the polyhedral angles are congruent: that is, that their face angles are equal respectively (which follows from the first part of the definition) and that their dihedral angles are equal respectively. The latter condition follows from the first part of the definition. It can be shown[1] that a polyhedron is determined by its faces. (This important theorem has, so far as I can learn, been omitted from American text-books.) This being the case, a polyhedron defined by the first condition is congruent to itself when any two vertices are taken as homologous to each other,

[1] Niewenglowski et Gerard, *Géométrie dans l'espace*, Paris, Gauthier-Villars, § 490.

and the definition should read: "a convex polyhedron whose faces are congruent regular polygons, the same number about each vertex."

Note.—The above reference to a polyhedral angle as a *configuration* suggests question 26. Is not the polyhedron also a *configuration*, and likewise all the other figures of solid geometry and of plane geometry? EDITOR.

CORRESPONDENCE.

TO THE EDITOR OF THE MONTHLY: In the review published in the March number of THE AMERICAN MATHEMATICAL MONTHLY, Professor E. J. Moulton writes, page 94,

> The "*and conversely*" is subsequently neglected without comment in deriving equations except in the case of the circle. . . . The proof of the "*and conversely*" for a straight line is as difficult as the direct, and the omission seems hardly excusable.

Since the proof of the "*and conversely*" for a straight line appears in the book under "review," Article 47, page 58, the reviewer is obviously in error. Moreover, the converse question for the several conics is considered in chapter VIII. That this question is "subsequently neglected" is therefore misleading.

As to the second criticism on "the equation of the locus," one may doubt the wisdom of introducing imaginary loci in an elementary text to the extent, at least, that would be necessary if one were to give a satisfactory explanation of the conditions under which $f = 0$ and $f \cdot \varphi = 0$ are equations of the same real locus.

L. WAYLAND DOWLING.

NOTES AND NEWS.

EDITED BY W. D. CAIRNS.

At Wellesley College Miss HELEN A. MERRILL has been promoted to a full professorship in mathematics. She is at present on leave of absence.

Professor F. A. SHERMAN, who in 1911 retired from the department of mathematics in Dartmouth College after forty years' service, died February twenty-fifth, 1915.

Professor THOMAS S. FISKE has been designated as administrative head of the Columbia University department of mathematics for two years beginning July 1, in the place of Professor Cassius J. Keyser, who retires at his own request.

The Mathematics Teacher has established a bureau for the use of institutions that need teachers and for the benefit of teachers of mathematics who wish to better their positions.

At the South Dakota State College, Assistant Professor CLIFFORD N. MILLS has organized a mathematical club for the undergraduate students. The members of the club are engineers and general science students majoring in mathematics.

Professor W. J. Hussey, of the department of astronomy in the University of Michigan, who has been for the past six months at La Plata University, has now returned to Ann Arbor.

School Science and Mathematics for April prints a valuable paper by C. W.

Newhall on "Recreations in secondary mathematics"; this gives both historical and topical treatment of the subject matter in a form which makes it quite accessible to the average high school teacher.

The address on "Graduate mathematical instruction for graduate students not intending to become mathematicians" delivered by Professor C. J. KEYSER before Section A of the American Association is printed in full in *Science* for March 26. His theme is the desirability and feasibility of organizing in every university a course such that an advanced student, while specializing in some other field, may yet gain a general knowledge of mathematical problems and processes.

The mathematics section of the Michigan Schoolmasters' Club met in Ann Arbor, April 1, 2 and 3. The program consisted of short discussions of practical phases of the teaching of high school mathematics, among which were "Practical applications of high school mathematics" and "Correlation between mathematics and other branches, and correlation of the various mathematical disciplines."

The Annual Conference of high school teachers in Kansas with the State University was held at Lawrence on March 26, 27, 1915. The leading paper at the mathematics section was given by Professor H. E. SLAUGHT, of the University of Chicago, on the topic: "Retrospect and Prospect in High School Mathematics." There was also an informal meeting of a dozen teachers of college mathematics, who were anxious to discuss matters of interest in the collegiate field.

The latest bulletin from the United States Bureau of Education on "Curricula in Mathematics" is a comparison of courses in the countries represented in the International Commission on the teaching of mathematics. The bulletin was prepared by J. C. Brown, of Teachers College, with the editorial coöperation of the American members of the commission.

The spring meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on April 2, 3, 1915. About seventy-five were in attendance upon the various sessions, including fifty-three members of the Society. There were twenty-seven papers presented, besides a report by Professor E. H. Moore, a former chairman of the Section, on "Integral equations in general analysis." A specially enjoyable feature of these meetings is the opportunity for social intercourse between the sessions and at the dinner on the evening of the first day, where many matters of mathematical interest are talked over informally.

In *School and Society* for March 27, 1915, Professor G. A. MILLER has an interesting article on "Shamelessness in regard to mathematical ignorance."

Superintendent BEN BLEWETT of St. Louis gave before the National Council of Education at its recent Cincinnati meeting, an address of appreciation of the late James M. Greenwood, whose interest in mathematics was noted in the February MONTHLY. A significant sentence in this address bears eloquent

testimony to the training power of mathematics and allied subjects which was so strongly manifest in his life: "Mathematics, logic, and philosophy sharpened an analytic mind which had singled them out as subjects of the greatest interest. Their influence on him was manifest in the comprehensive grasp and ordered marshalling of his thoughts, and in the clear and vigorous English in which he expressed them."

The University of Wisconsin, in coöperation with the University High School of Madison, has instituted a scheme for supervised observation work as a part of practice teaching. To avoid having ineffective teaching by beginners and nevertheless to give these proper training, they are enrolled as regular members of the class, sharing in all the recitations and other responsibilities of the pupils, but are detailed frequently to act as assistant teachers in various capacities. They must further report upon various problems of teaching as these arise, all such reports being discussed with them by the supervisors and recorded in the principal's office where they may be consulted by school superintendents seeking teachers.

The March number of *The Mathematics Teacher* contains the following four articles: (1) "The five Platonic bodies," by J. H. Weaver, in which he points out mutual relations of these bodies and of thirteen other solids studied by Archimedes, and argues that these were all derived by being whittled from the sphere; (2) "A study of the reliability of test questions," by G. G. Chambers, an account of preliminary work to establish non-geometrical tests for determining the results of geometry teaching on reasoning ability; (3) "Business arithmetic versus algebra in the high school," an address by G. H. Van Tuyl of the High School of Commerce, New York City, in support of the former, supplemented by a discussion from a different point of view by W. S. Schlauch; and (4) the report of the arithmetic committee of the Association of Teachers of Mathematics in the Middle States and Maryland. The spring meeting of this association was held at Hunter College, New York City, on April 17.

The *Mathematical Gazette* is presenting, in a form adapted to the use of school teachers, a series of articles by Professor W. W. Rouse Ball on the world's great mathematicians. The following abstract of the article on Pythagoras in the January number will furnish some idea of the nature of this valuable series of biographical sketches.

To the critical historian it appears to be established that Pythagoras was born about 570 B.C., that through his father's trade relations he came into touch with philosophers of the Ionian School and later went to Egypt where, through favorable introductions, he secured admission to the College of Priests and mastered the secrets of their science and religion. Thence he went to Babylon, where he learned something of Persian and Indian thought at the very time, it may very well be, when Daniel of the Old Testament was living there. While it seems certain that he was influenced by the geometrical discoveries of the Ionian School, which included the earliest attempts to give general, rather than

particular, proofs for geometrical propositions, he was more influenced by the systems of religion, politics, and philosophy which he found in Egypt and Babylon. He was indeed a moral teacher and philosopher preëminently, yet to him is due the organization of mathematics as a science, even the adoption of the word itself.

Ball's theory is that Pythagoras was led to the study of geometry and numbers through his researches in natural philosophy. His love for music brought him to his important study of vibrating strings and this theory in turn he naturally connected with the properties of numbers. From this starting point too he went on to develop very elaborately the connection between the properties of numbers and various recurrent phenomena, geometrical forms, the nine celestial bodies (the earth, the moon, the sun, the five planets then known, and the firmament), etc. Just because of this (to him) essential and universal interrelation, he organized and extended the previously known portion of what we call elementary geometry, he discovered and proved the existence of incommensurable quantities but treated these by geometrical methods, distrusting such demonstrations as rest on the possibility of making numerical measurements. He was particularly concerned with the development of the properties of integers, being certainly acquainted with triangular numbers, excessive, perfect, and defective numbers, amicable number pairs, arithmetic, geometric and harmonic series, etc. How fundamental in the mind of Pythagoras were number theory and geometry to a knowledge of higher learning, may be judged from this, that the *quadrivium*, composed of numbers absolute or Pythagorean arithmetic, numbers applied or music, magnitudes at rest or geometry, and magnitudes in motion or astronomy, formed his great scheme of liberal education preliminary to any further pursuit of culture and philosophy.

Because he let his imagination run away with him and his theories were mixed with fanciful speculations, which were often based on inferences rather than on exact observations, he and his school were subjected to severe criticism and mistrust both in ancient and in modern times; but it is now acknowledged that he initiated the brilliant era of Greek philosophy and science, and that, even though his political activity aroused much opposition in his lifetime, the loftiness of his aims and ideals gives him an unquestioned high place.

Errata.—The following errors have been discovered: Volume XXI, page 299, last line, $(-1)^k$ should be $(-1)^{k-1}$. Page 323, line 10, 1656 should be 1856. Page 333, line 7 down, delete the word "nine"; line 7 up, $x^9 - 2x^6 - x^3 + 1$ should be $x^9 - 2x^6 - x^3 - 1$; line 4 up, $(-1)^{13/9}$ should be $(-1)^{12/9}$. Page 334, line 18 down, $\gamma + 2C_2$ should be $\gamma + 1C_2$. Page 342, line 3 down, the fifth value should be 8 instead of 5. Volume XXII, page 68, line 13 down, $f''(x) = -e(2n-1)\pi$ should be $f''(x) = e^{-e(2n-1)\pi}$. Page 131, line 4, $\pi^3/12^3$ should be $\pi^3/12$.

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HISTORY OF ZENO'S ARGUMENTS ON MOTION:

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

By FLORIAN CAJORI, Colorado College.

VII.

7. KANT AND OTHER PRE-CANTORIAN DISCUSSION.

We now come to a commanding figure in philosophic thought—Emmanuel Kant. He took Zeno's dialectics more seriously than had been the custom before. Kant says that critics charged Zeno with a complete denial of both of two self-contradictory propositions. "But," says Kant, "I do not think that he can be rightly charged with this."[1] Zeno was not as much of a skeptic as has been pretended. Kant did not write on Zeno's arguments on motion, but he touched on other arguments of Zeno. Kant's first antinomy, or "the first conflict of the transcendental ideas," contains parts which remind one of the following annihilation of the notion of space, as given by Zeno: If there is space, it is in something, for every thing that is, is in something; but that which is in something, is also in space. Space, then, must also be in space, and so on infinitely: therefore there is no space. While Kant did not contribute directly to a clearer understanding of Zeno's arguments on motion, the effect of his writings was a more painstaking and searching examination of that subject.

In 1794 there appeared in Halle a monograph on Zeno's arguments on motion by C. H. E. Lohse, which is permeated by the atmosphere of Kantian philosophy. It is the earliest publication on our topic which appeared in the form of a monograph.[2] Of its four parts, the first deals with Zeno's system in general, the second gives his arguments against motion, the third elucidates Aristotle's refutation of Zeno, the fourth deals with "the only way" of refuting Zeno. The last argument, the "stade," is not discussed at all. Aristotle's distinction between a potential

[1] *Kant's Werke*, Bd. III, "Kritik der reinen Vernunft," 2. Aufl. (1787), Berlin, 1904, p. 345.

[2] Car. Henr. Erdm. Lohse, *Diss.* (*praeside Hoffbauer*) *de argumentis, quibus Zeno Eleates nullum essē motum demonstravit et de unica horum refutandorum ratione.* Halle, 1794. All our information on Lohse's paper is taken from E. Wellmann, *op. cit.*, pp. 12–14.

and an actual division to infinity is pronounced arbitrary. Whatever can be divided to infinity, says Lohse, actually consists of an infinite number of parts which exist even before division. He decides on this point against Aristotle and in favor of Zeno, as Bayle had done, though he does not mention Bayle. Lohse claims that Zeno's fundamental error lay in a wrong conception of time and space. These are not qualities subject to our senses, but are forms which determine the manner in which our senses are affected; they are *a priori* ideas. Time and space can both be divided to infinity, but one cannot consider time as made up of indivisible points in the manner of Zeno, else what happens in a moment of time would happen in no time. Rest is not, as Zeno and his followers claim, the absence of motion; it is the least velocity of succession. A body can be perceived only as it moves. "Without doubt," says Lohse, "all mistakes of their system sprang from that error. Thence it came that reason and the senses seemed to contradict each other."

Presiding at the time when Lohse presented his dissertation for an academic degree at Halle was Joh. Christoph Hoffbauer (1766–1827) who, many years later, prepared a cyclopædia article, "Achilles (Der Trugschluss)."[1] After expressing his disapproval of Facciolati's argument (previously referred to) he states that Zeno's argument is true only on a condition which has not been stated explicitly: Zeno's contention that the faster runner will always only arrive at the places where the slower has been, and will be behind the slower runner, is true only on condition that the faster has not overtaken the slower. The only thing proved by Zeno is therefore that the faster runner cannot have overtaken the slower as long as the slower is still in advance!

A reply to Hoffbauer's argument was made by Christian Ludwig Gerling, professor of mathematics, astronomy, and physics at the University of Marburg, in a prorectorat address.[2] The claim that Zeno's argument is valid only for certain points, not for all, is no objection at all, unless it is first shown to be a mistake to assert as true for all points what is in fact true of an infinite number of points; a defender of Zeno may always demand that the points be shown, for which the proof does not hold. Gerling insists that Hoffbauer himself reasons in a circle when he accuses Zeno of reasoning in a circle, for whoever has still to prove the possibility of an overtaking is not yet permitted to speak of the time before or after which the overtaking takes place.

Against Waldin's argument, advanced at this same university (Marburg) forty-three years previous, to the effect that Zeno assumes the existence of motion, the very thing that is in dispute, Gerling argues that Zeno's argument is an indirect one, a *reductio ad absurdum*, the form of which is quite valid.

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[1] *Allg. Encycl. d. Wissensch. u. Künste*, von J. S. Ersch u. J. G. Gruber, Leipzig, 1818.

[2] *De Zenonis Eleatici paralogismis motum spectantibus, Dissertatio auctore Chr. Lud. Gerling.* Marburg, 1825. We know this dissertation only from the description of it given by E. Wellmann, *op. cit.*, pp. 14, 15, and by Dr. Johann Heinrich Loewe, "Ueber die Zenonischen Einwürfe gegen die Bewegung," in *Böhm. Gesellsch. d. Wissensch.*, VI Folge, 1 Bd., 1867, pp. 30, 34. In Poggendorff's *Handwörterbuch*, the date of Gerling's dissertation is given as 1830. We have seen references to an edition in German of the year 1846. From this we infer that several editions of it have appeared, and that it enjoyed a considerable circulation.

Lohse's metaphysical apparatus Gerling declares needless and useless. In the constructive part of his dissertation, Gerling dwells on the distinction between continuous and discrete quantity, admits the infinite divisibility of space and time, and constructs the infinite geometric progressions whose sums give respectively the distance and the time of running, before Achilles overtakes the tortoise. Gerling here repeats what Gregory St. Vincent had done long before, only Gerling uses letters, while Gregory assumed a special numerical case. Gregory is nowhere mentioned by Gerling. The sums of the two geometric progressions are values which in no way conflict with the estimate obtained from sensuous perception; Zeno's paradox, as interpreted by aid of the mathematical formulas, conflicts in no way with experience. Hence the puzzle is solved. Though a mathematician, Gerling does not feel the need of explaining the possibility of a variable reaching its limit.

As to the "Arrow" a sharp distinction between the continuous and the discrete is sufficient. In continuous quantity the number of possible subdivisions is *arbitrary*, and each subdivision is itself continuous. Hence Zeno's alleged denial of the infinite divisibility does not follow. Gerling treats the "Stade" with more than customary respect, and admits that, if one assumes with Zeno that space and time be not infinitely divisible, then it follows, as Zeno says, that half the time is equal to the whole time.

An entirely different type of discussion, more along the lines of Kant, profound and obscure, is given by Georg Wilhelm Friedrich Hegel. He holds the view that "Zeno's dialectic of matter has not been refuted to the present day; even now we have not got beyond it, and the matter is left in uncertainty."[1] He protects Aristotle against Bayle who objected to Aristotle's distinction between a potential and an actual subdivision of a line to infinity. Hegel keenly realizes the speculative importance of Zeno's paradoxes and points out that the dialectician of Elea had analyzed our concepts of time and space and had pointed out the contradictions involved therein; "Kant's antinomies do no more than Zeno did here."[2] Movement appears "in its distinction of pure self-identity and pure negativity, the point as distinguished from continuity."[3] This continuity is an absolute hanging together, an annihilation of all differences, of being by itself; the point on the other hand is pure existence by itself, the absolute distinctness from others, the suspension of all self-identity and all hanging together. In time and space the opposites are united in one, hence the contradiction as exhibited in motion. Hegel's position is a long way, still, from Georg Cantor's continuum, with its skilful union of continuity and discreteness. In the "dichotomy" the assumption of half a space is incorrect, says Hegel, "there is no half of space, for space is continuous; a piece of wood may be broken into two halves, but not space, and space only exists in movement."[4] Motion is connectivity, disintegration into an indefinite number of aggregates is its opposite.

[1] G. W. F. Hegel, *History of Philosophy*, transl. by E. S. Haldane, Vol. I, London, 1892, p. 265. See also Hegel's *Sämtl. Werke*, Bd. 13, 1833, pp. 314–327.

[2] Hegel [ed. Haldane], Vol. I, p. 277.

[3] Hegel, *op. cit.*, Vol. I, p. 268.

[4] Hegel, *op. cit.*, Vol. I, p. 271.

Somewhat more specific and comprehensible are the ideas set forth by his philosophical opponent, Johann Friedrich Herbart. Zeno's paradoxes are taken up by him in two works, his popular *Einleitung in die Philosophie* (1813) and his more technical and scientific *Allgemeine Metaphysik* (1828–9). Only in the latter work is the solution of the contradictions attempted. From it we quote:[1]

"The argument inevitably confuses those, who admit the infinite divisibility of the path and then console themselves with a corresponding infinite divisibility of the time, to such a degree that though at first willing to consider the process of dividing, which must continue to infinity, they soon in one leap consider the infinite number of time intervals as passed over, since they see that they must combine the infinite number of parts of the time as well as of the path to the place of overtaking, which they cannot do. The leap and the doubly infinite division are both faulty and amount to naught."

Thus, this infinite subdivision of the time and space is rejected by Herbart, because the mind is not able to *imagine* all the steps in the process. Imaginability is made the criterion of truth or error. This criterion throws out infinity at once; it throws out non-euclidean geometry and other parts of mathematics. We cannot really imagine things which we have never seen. Our senses are inaccurate, our intuitions are crude; hence it would seem to us impossible to build up sound mathematical theory, if everything unimaginable were to be cast aside. Herbart tries to explain motion by the concept of *velocity*, which seems itself to involve a contradiction, that Herbart endeavors to resolve by his theory of a "rigid line," a sort of continuum, which might have given rise to great possibilities upon more careful development. As it is, it offers greater obstacles by far than does the original "Dichotomy" or "Achilles" which it is intended to explain.

A still different attitude toward Zeno's paradoxes is taken by Friedrich Adolf Trendelenburg of the University of Berlin, in his *Logische Untersuchungen*, 1840, where he constructs his philosophic system upon the concept of motion. Constructive motion is common to the external world of being and to the internal world of thought, so that thought, as the counterpart of external motion, produces from itself space, time, and the categories. Motion is undefinable. In accordance with this view it is only through motion that Zeno's arguments against motion have come to be. For they depend upon the division of time and space, and the synthesis of those divisions. But division and synthesis are nothing but special forms of motion. What the proofs combat, they themselves use as the means of combat, and thereby testify to the controlling nature of motion. Trendelenburg and Kant evidently begin at opposite ends; Kant takes time and space as *a priori* ideas, and motion as secondary and dependent upon them; Trendelenburg makes motion the *a priori* idea, and pretends to derive time and space from it.

Friederich Ueberweg of the University of Königsberg[2] refers to our subject in different parts of his *Logik*. He says in one place that the "Achilles" proves too little; it proves merely that the tortoise cannot be overtaken within a definite series, and then claims that the tortoise cannot be overtaken anywhere and at

[1] J. T. Herbart, *Sämtl. Werke*, herausgeg. von Karl Kehrbach, Langensalza, Vol. VIII, 1893, p. 177.

[2] F. Ueberweg, *System der Logik*, 2. Aufl., p. 387 ff.

any time. True as this criticism may be, it does not illuminate the matter sufficiently to satisfy the reader.

Of the same type, but fuller in statement, is a criticism by Carl Prantl,[1] professor at the University of Munich. He claims that Zeno discarded the concept of continuity by considering only some particular points on a line and only some particular moments in time. By drawing his inferences from these disintegrated fragments of time and space, Zeno was able to advance contradictions in a picturesque manner. This conversion of the general and continuous into the particular and momentary will be encountered often, says Prantl, in those who care more for rhetorical form than for true philosophy.

Much confidence in his ability to clear up the mystery of Zeno's paradoxes is displayed by Eugen Carl Dühring in his *Kritische Geschichte der Philosophie*, first edition, 1869. Three concepts are necessary here: rest, motion and position. Usually only the first two are considered. At each moment (point) of time a moving body has a definite position but no motion. This fact makes it difficult to explain motion. He says further:[2] "The compelling force and real conclusiveness of the Eleatic contentions is to be found chiefly and almost exclusively in the logical necessity which does not permit the infinite to be thought of as completed, as enumerated so to speak, and concluded. . . . It is the concept of infinity which proves itself everywhere and also where it is not readily recognized, as the true cause of the contradictions." Dühring discusses infinity in several places of his works. He believes in the infinity usually set forth in the study of the calculus,—a variable which increases without limit, but at any moment has really a finite value. He makes war against the concept of an actual infinity—"jene wüste, sich widersprechende Unendlichkeit." "The infinite divisibility indicates . . . only this, that I can conceive the division of a quantity as far as I choose, without limit. If, on the other hand, I consider the division to infinity as really existing outside of my presentation of it, then there soon result the most manifold contradictions. . . . As regards motion, it must be recognized that it belongs to the empirical concepts, i. e., in our thinking there remains here always an unrecognizable residue, for we must give up the attempt to penetrate to the reasons of the phenomena." Georg Cantor criticizes Dühring in these words:

"The proofs of Dühring against the properly-infinite could be given in much fewer words and appear to me to amount to this, either that a definite finite number, however large it may be thought to be, can never be an infinite number, as follows immediately from the concept of it, or else that the variable, an unlimitedly large finite number, cannot be thought of with the quality of definiteness and therefore not with the quality of existence, as follows again from the nature of the variability. That not the least is hereby established against the conceivability of transfinite number, I feel certain; and yet, those proofs are taken as proofs against the reality of transfinite numbers. To me this mode of argumentation appears the same as if, from the existence of innumerable shades of green, we were to conclude that there can be no red."[3]

Dühring's explanation of infinity and of Zeno is accepted by Eduard Wellmann, in his historical monograph[4] of 1870. Another research, partly historical

[1] Carl Prantl, *Geschichte der Logik im Abendlande*, 1. Bd., Leipzig, 1855, pp. 10, 11.

[2] *Kritische Gesch. d. Philosophie*, Dr. E. Dühring, Leipzig, 1894, p. 49.

[3] Georg Cantor, *Grundlagen einer allg. Mannichfaltigkeitslehre*, Leipzig, 1883, p. 44.

[4] E. Wellmann, *op. cit.*, p. 23.

and partly expository, was published in 1867, by Johann Heinrich Loewe, a pupil of the philosopher, Anton Günther of Vienna. It is referred to by Knauer[1] as the most acute and satisfactory explanation that has yet been offered. "The solution of the riddle," says Loewe,[2] "appears to us to lie in the knowledge that contradictions must arise inevitably, as soon as space, time, and motion are considered at the same time from the stand-point of sensuous presentation and of non-sensuous conceptual reasoning." One point of view appeals to the imagination; the other to abstract thought. Sensuous perception can follow the process of infinite division only a little way, everything beyond is a matter of pure reason. Gerling's presentation of "Achilles" is an appeal to reason. As long as one considers the infinite multiplicity of small distances and of time-intervals, one approaches the riddles from the standpoint of abstract thought; when one appeals to the imagination, then the finite time and the finite length of the race stand out. Loewe seems still to hold to the old view that thought can recognize no end to a motion which extends over an infinite process. Hence the contradiction must stand, the antinomy is evident.

Thus we see that German philosophy down to the last quarter of the nineteenth century continually accentuates the existence of contradictions in the problem of motion.

Some English thinkers of the nineteenth century, who were interested in Zeno's arguments, came under the influence of Kantian philosophy. The Kantian attitude toward Zeno is described in the article "Zeno" in the eighth edition of the *Encyclopaedia Britannica* (1860) thus:

> "He brought a most powerful mind to his task, and, curious to say, subsequent thinkers have very generally agreed in misunderstanding both his reasoning and his method, and it is only of late years that Kant, in his *Antinomies of the Pure Reason* (see *Kritik der Reinen Vernunft*) seized upon the much maligned doctrines of the Eleatic, and held them up to the admiration of all true thinkers as rare examples of acute and just thought. Bayle, in a clever paper on Zeno, in his *Dictionnaire*, makes him, according to custom, a sceptic. Brucker finds that Zeno surpasses his intelligence, and he is content to make him a pantheist. Others again, have charged him with nihilism. Zeno, fortunately, can afford to sit quite easy to all those affronts offered to his reason . . . they [arguments against motion] all take their rise, as Kant and Hamilton (*Lectures on Metaphysics*) have shown, from the inability of the mind to conceive either the ultimate indivisibility, or the endless divisibility, of space and time, as extensive and as protensive quantities. The possibility of motion, however certain as an observed fact, is thus shown to be inconceivable. To have discovered this peculiarity of our mental constitution, and to have stated it with eminent clearness, belongs to Zeno the Eleatic, and to him alone."

Sir William Hamilton puts this matter thus:[3]

> "Time is a protensive quantity, and, consequently, any part of it, however small, cannot, without a contradiction, be imagined as not divisible into parts, and these parts into others *ad infinitum*. But the opposite alternative is equally impossible; we cannot think this infinite division. One is necessarily true; but neither can be conceived possible. It is on this inability of the mind to conceive either the ultimate indivisibility, or the endless divisibility of space and time, that the arguments of the Eleatic Zeno against the possibility of motion are founded,—arguments which at least show, that motion, however certain as a fact, cannot be conceived pos-

[1] Vincenz Knauer, *Die Hauptprobleme der Philosophie*, Wien u. Leipzig, 1892, p. 54.

[2] J. H. Loewe, *op. cit.*, p. 32.

[3] *Lectures on Metaphysics and Logic*, by Sir William Hamilton, Vol. I, Boston, 1863, Lecture 38, p. 530.

sible, as it involves a contradiction. . . . Now the law of mind, that the conceivable is in every relation bounded by the inconceivable, I call the Law of the Conditioned."

John Stuart Mill, in his *Logic*,[1] refers to Thomas Brown who considered the "Achilles" insoluble, and then offers a solution to the invention of which he lays no claim. It presents no new points of view. Herbert Spencer discusses questions of time and space in his *First Principles*[2] and concludes in general that "ultimate scientific ideas, then, are all representative of realities that cannot be comprehended." In particular, "halve and again halve the rate of movement for ever, yet movement still exists; and the smallest movement is separated by an impassable gap from no movement."

It is readily seen that the nineteenth century philosophers had penetrated deeper than most of their predecessors and had encountered difficulties previously neglected by Hobbes and others who seemed to think that they had solved the "Achilles" paradox by the mere statement that time, as well as space, was infinitely divisible. What came to be thoroughly realized since the time of Kant was the impossibility of *imagining* the "Achilles" from the standpoint of infinite divisibility of a distance, that all appeals to intuition were futile. When Spencer says that infinite divisibility cannot be "comprehended," and Thomas Brown and Sir William Hamilton say that motion is "insoluble" and "inconceivable," I take it that they mean simply that these processes are *unimaginable*, that they are beyond the reach of our sensual intuitions. I do not interpret them to mean that these processes are beyond the reach of logic, beyond the reach of the reasoning faculty so as to be, and forever remain, wholly mysterious. Mathematics includes among its results numerous teachings which one cannot "imagine." Probably no one claims to be truly able to visualize to himself the non-euclidean geometries; analysts do not claim to be able to imagine or see a continuous curve which has no tangent line at any of its points. Yet no modern mathematician rejects non-euclidean geometries and non-differentiable continuous curves.

These unimaginable mathematical creations are admitted into the science as a matter of necessity. Felix Klein states the issue as follows: "As the subjects of abstract geometry cannot be sharply comprehended through space intuition, one cannot rest a rigorous proof in abstract geometry upon mere intuition, but must go back to a logical deduction from axioms assumed to be exact."[3]

It so happens that England's two famous opium eaters, Thomas De Quincey and Samuel Taylor Coleridge, were interested in the "Achilles." Coleridge's critical powers were set forth by De Quincey in the following terms:[4]

"I had remarked to him that the sophism, as it is usually called, but the difficulty, as it should be called, of Achilles and the Tortoise, which had puzzled all the sages of Greece, was, in fact, merely another form of the perplexity which besets decimal fractions; that, for example, if

[1] *A System of Logic*, Vol. II, London, 1851, p. 381.

[2] H. Spencer, *First Principles of a New System of Philosophy*, New York, 1882, pp. 47–67.

[3] F. Klein, *Anwendung der Differential- und Integralrechnung auf Geometrie*. Leipzig, 1907, p. 19.

[4] *Tait's Magazine*, Sept. 1834, p. 514.

you throw $\frac{2}{3}$ into a decimal form, it will never terminate, but be .666666, etc., *ad infinitum*. 'Yes,' Coleridge replied, 'the apparent absurdity in the Grecian problem arises thus,—because it assumes the infinite divisibility of space, but drops out of view the corresponding infinity of time.' There was a flash of lightning, which illuminated a darkness that had existed for twenty-three centuries."

As a matter of fact, Aristotle had seen that far. But Coleridge proceeded somewhat farther in an essay on Greek sophists in *The Friend*,[1] where he says:

"The few remains of Zeno the Eleatic, his paradoxes against the reality of motion, are mere identical propositions spun out into a sort of whimsical conundrums, as in the celebrated paradox entitled Achilles and the Tortoise, the whole plausibility of which rests on the trick of assuming a *minimum* of time while no *minimum* is allowed to space, joined with that of exacting from *intelligibilia*, *νούμενα*, the conditions peculiar to objects of the senses *φαινόμενα* or *αἰνϑανόμενα*."

What belongs to Coleridge himself in this passage is the contention that the sophism consists in applying to an idea conditions only properly applicable to sensuous *phaenomena*. Coleridge's argument was elaborated many years later in dialogue form, by Shadworth H. Hodgson. We give the critical part of the discussion:[2]

". . . being infinitely *divisible* is not the same thing as being infinitely *divided*. Actually to divide to infinity that hundredth part of a minute, in which (*phenomenally* as you say) Achilles overtakes the tortoise, is an infinitely long operation. . . . And *this* division you call upon Achilles to perform, before the tortoise can be overtaken, and to perform *phenomenally*. . . . You require that Achilles shall exhibit to the senses the infinite divisibility of time and space, which appertains to them truly indeed, but only as objects of imagination and thought. . . . The world of *thought* and *reality* is not a world apart, but is identical with the phenomenal world, only differently treated. . . . Neither is there any contradiction between them. Phenomenal motion is as infinitely divisible in *thought* as time and space are."

This explanation does not explain. Even as "objects of the imagination" the infinite divisibility of time and space is a source of perplexity. Our imagination is unable to follow Achilles to the end, through the infinities of time and space intervals. Moreover, "*thought* and *reality*" are indeed worlds "apart" whenever the time intervals, corresponding to the space-intervals passed over by Achilles, are so taken that they form together an infinite series that is *divergent*, so that, in thought, Achilles never overtakes the tortoise; in Zeno's traditoinal argument, "thought and reality" were "apart."

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[1] *Complete Works of S. T. Coleridge*, Vol. II, New York, 1856, p. 399.
[2] *Mind*, London, Vol. V, 1880, pp. 386–388.

[The remaining parts of this series are: *D*. Viewed in the Light of an Idealistic Continuum (G. Cantor); *E*. Post-Cantorian Dissension.]

LINEAR MOMENTUM, KINETIC ENERGY, AND ANGULAR MOMENTUM.

By E. B. WILSON, Massachusetts Institute of Technology.

The MONTHLY for December, 1914, contains an interesting and important note by E. V. Huntington on uniplanar rigid motion in which it is pointed out that the relation between angular acceleration and moments of forces is incorrectly given in a number of texts and in which the correct relation is clearly exhibited. We may point out that the theorem on energy is also at fault in some books. For instance, in Smith and Longley's *Theoretical Mechanics*, page 234, we find that the kinetic energy of a lamina moving in its own plane is the sum of two terms:

$$\text{K. E.} = \tfrac{1}{2}Mv_0^2 + \tfrac{1}{2}I_0\omega^2, \qquad \text{(A)}$$

where v_0 is the speed of any point of the lamina, I_0 the moment of inertia of the lamina about that point, and ω the angular velocity which is the same about all points.

This remarkably untrue result is of course "proved," not merely stated. The error is more grievous than that on the angular acceleration, because to err in a first order effect (a velocity) is far less excusable than to err in a second order effect (an acceleration). The knowledge that plane motion is *kinematically* a composite of rotational velocity about any point and a translational velocity of that point is so ingrained in the minds of some persons that they do not recognize that *dynamically* the composition is invalid. It is not that the right results are any harder to prove than the wrong ones; it is merely evidence in high quarters of the same sort of intellectual inattention that leads the sophomore or junior to "solve" all problems in rectilinear motion by his familiar formulas for uniform motion, however much the motion may be irregular.

We propose to give, in as direct a manner as possible, a correct treatment of linear momentum, kinetic energy, and angular momentum, whether they be measured absolutely or relative to a moving origin. (There will be no reference to moving *rotating* axes, which introduce comparatively difficult and delicate considerations.) We shall use a minimum amount of vector analysis, following the notations of Gibbs, to avoid writing the equations for each component separately. Let us denote by

$\mathbf{r}$, absolute position of any point P, $\quad$ $\mathbf{v}$, absolute velocity of P,
$\mathbf{s}$, position of moving origin O, $\quad$ $\mathbf{u}$, velocity of moving origin O,
$\mathbf{r}'$, position of P relative to O, $\quad$ $\mathbf{v}'$, velocity of P relative to O.

(Absolute position and velocity mean position and velocity referred to a fixed origin; we mean no metaphysics.) We then have the equations

$$\mathbf{r} = \mathbf{s} + \mathbf{r}', \quad \mathbf{v} = \mathbf{u} + \mathbf{v}'. \qquad (1)$$

The second is the "parallelogram law" for compounding velocities; the first states that one vector side of a triangle is the sum of the other two sides as vectors.

Linear Momentum. Let particles of mass m_i ($i = 1, 2, \cdots$) be situated at the points P_i. By definition the linear momenta of each individual mass and of the whole set of masses are respectively

$$m_i\boldsymbol{v}_i \quad \text{and} \quad \text{L. M.} = \sum_i m_i\boldsymbol{v}_i. \tag{2}$$

To simplify the second of these statements we introduce an ideal point G, called the center of mass or the center of gravity, by the definition

$$(\sum_i m_i)\bar{\boldsymbol{r}} = \sum_i m_i\boldsymbol{r}_i \quad \text{or} \quad M\bar{\boldsymbol{r}} = \sum_i m_i\boldsymbol{r}_i, \tag{3}$$

where $\bar{\boldsymbol{r}}$ is the absolute position of G, and M is the total mass of the system. If we differentiate with respect to the time and denote by $\bar{\boldsymbol{v}}$ the velocity of G, we have

$$M\bar{\boldsymbol{v}} = \sum_i m_i\boldsymbol{v}_i = \text{L. M.} \tag{4}$$

We may therefore state

THEOREM 1. *The linear momentum of a system of particles is equal to the product of the mass of the system and the velocity of its center of gravity.*

By substitution of (1) in (2) and reduction we have

$$\text{L. M.} = \sum_i m_i\boldsymbol{v}_i = \sum_i m_i\boldsymbol{u} + \sum_i m_i\boldsymbol{v}_i' = M\boldsymbol{u} + \sum_i m_i\boldsymbol{v}_i'.$$

Now $\Sigma m_i\boldsymbol{v}_i'$ is the relative linear momentum, by definition. Hence

THEOREM 2. *The linear momentum of a system is the sum of the relative linear momentum and of the momentum of a mass equal to the total mass of the system moving with the point of reference.*

THEOREM 3. *The relative linear momentum $\Sigma m_i\boldsymbol{v}_i'$ is equal to $M\bar{\boldsymbol{v}}'$, the product of the mass and the relative velocity of the center of gravity.*

The proof of Theorem 3 is left to the reader.

Kinetic Energy. By definition, the kinetic energies of each mass and of the whole set of masses are respectively

$$\tfrac{1}{2}m_i\boldsymbol{v}_i^2 \quad \text{and} \quad \text{K. E.} = \tfrac{1}{2}\sum_i m_i\boldsymbol{v}_i^2, \tag{5}$$

it being agreed that by the square of a velocity we mean the square of its magnitude, or, what amounts to the same thing, the scalar product of the vector velocity into itself,—thus $\boldsymbol{v}^2 = \boldsymbol{v}\cdot\boldsymbol{v}$.[1]

On substitution of (1) in (5) and reduction we find

$$\text{K. E.} = \tfrac{1}{2}\sum_i m_i\boldsymbol{v}_i\cdot\boldsymbol{v}_i = \tfrac{1}{2}\sum_i m_i(\boldsymbol{u} + \boldsymbol{v}_i')\cdot(\boldsymbol{u} + \boldsymbol{v}_i')$$

[1] The object of introducing the scalar product is to render the square of a vector which may be written as the sum of two vectors amenable to the ordinary distributive laws of multiplication; we may thereby avoid the use of the law of cosines.

$$= \tfrac{1}{2}\sum_i m_i \boldsymbol{u}\cdot\boldsymbol{u} + \sum_i m_i \boldsymbol{u}\cdot\boldsymbol{v}_i' + \tfrac{1}{2}\sum_i m_i \boldsymbol{v}_i'^2.$$

Or

$$\text{K. E.} = \tfrac{1}{2}M\boldsymbol{u}^2 + \boldsymbol{u}\cdot(M\bar{\boldsymbol{v}}') + \tfrac{1}{2}\sum_i m_i \boldsymbol{v}_i'^2, \tag{6}$$

when by Theorem 3 we have replaced $\Sigma m_i \boldsymbol{v}_i'$ by $M\bar{\boldsymbol{v}}'$.

THEOREM 4. *The kinetic energy of a system is the sum of* THREE *terms: (a) the relative kinetic energy* $\frac{1}{2}\Sigma m_i \boldsymbol{v}_i'^2$, *(b) the kinetic energy of the whole mass located at the moving point of reference* $\frac{1}{2}M\boldsymbol{u}^2$, *(c) the product of the mass M by the scalar product* $\boldsymbol{u} \circ \bar{\boldsymbol{v}}'$ *of the velocity of the point of reference and the relative velocity of the center of gravity.*

The third term (c), which is $M\boldsymbol{u} \cdot \bar{\boldsymbol{v}}'$, vanishes only when: ($\alpha$) the point of reference O is (at least momentarily) at rest, or (β) the center of gravity is (at least momentarily) at rest relative to O, or (γ) the velocity of the point O and the relative velocity of the center of gravity are perpendicular.

To compute the kinetic energy of a rigid lamina we *do not need* to use Theorem 4; we may apply the original definition (5). For if the lamina has a motion of pure translation (at the instant considered), we find by direct summation that K. E. $= \frac{1}{2}M\boldsymbol{v}^2$, where $\boldsymbol{v}$ is the velocity of any point; whereas if the lamina is rotating, the velocity of any point is $r\omega$ in magnitude, r being the distance from the instantaneous center to the point, and we again can sum directly to find K. E. $= \frac{1}{2}I_C\omega^2$, where $I_C = \Sigma m_i r_i^2$ is the moment of inertia with respect to the instantaneous center.

In many cases it is *desirable* to express the kinetic energy of the lamina with respect to a point O fixed in the lamina. Then Theorem 4 is applicable, and the term (a) may be simplified by using $r'\omega$ as the magnitude of $\boldsymbol{v}'$, the velocity relative to the fixed point. We have therefore

$$\text{K. E.} = \tfrac{1}{2}M\boldsymbol{u}^2 + M\boldsymbol{u}\cdot\bar{\boldsymbol{v}}' + \tfrac{1}{2}I'\omega^2, \tag{6'}$$

where $I' = \Sigma m_i r_i'^2$ is the moment of inertia relative to the point O. Moreover, if θ be the angle between the directions $\boldsymbol{u}$ and $\bar{\boldsymbol{r}}'$, the angle between $\boldsymbol{u}$ and $\bar{\boldsymbol{v}}'$ is $90° + \theta$, and as the magnitude of $\bar{\boldsymbol{v}}'$ is $r'\omega$, we have

$$\text{K. E.} = \tfrac{1}{2}M\boldsymbol{u}^2 - Mu\omega\bar{r}'\sin\theta + \tfrac{1}{2}I'\omega^2. \tag{6''}$$

The kinetic energy consists of *three* terms and does *not* reduce to $\frac{1}{2}M\boldsymbol{u}^2 + \frac{1}{2}I'\omega^2$ unless $u = 0$, $\omega = 0$, $\bar{\boldsymbol{r}}' = 0$, or $\sin\theta = 0$.

THEOREM 5. *The proposition (A), that the kinetic energy of a lamina is the sum of two terms* $\frac{1}{2}M\boldsymbol{v}_O^2 + \frac{1}{2}I_O\omega^2$, *representing the kinetic energy of translation and of rotation with reference to some point O, as if kinematic resolution were dynamically right, is true for all points O upon the circle which at the instant considered contains the instantaneous center and the center of gravity as diametrically opposite points, and for no points not on this circle except when there is no rotation and the construction breaks down.*

This pretty little geometric result is probably not new, but I have no re-

membrance of having seen it. The proof, which follows immediately from our work just above, will be left to the reader.

Example. Let a hoop of mass m, uniformly distributed, and of radius a be loaded at one point of the rim with a mass m'. Let the angular velocity be ω, and the inclination of the radius from the center to m' be φ (measured backward from the downward vertical). Find the kinetic energy.

We may solve in three different ways: (α) by using the instantaneous center, (β) by taking as point of reference the center of the hoop, (γ) by taking as point of reference the center of gravity. The details of the work will be left to the reader, who will find that the result obtained from (β) will check with the others only when three terms are used in the expression for the kinetic energy.

The following are well-known propositions of kinematics in the case of rigid bodies in space: (a) motion about a fixed point is at each instant rotation about some axis through that point, (b) motion in general may be regarded as composed of translation of an arbitrary point fixed in the body and rotation about some axis through that point, (c) for all the points of the body the directions of the axes are the same, (d) there is one of these axes such that the translation of every point on it is along the axis. It is by this suite of theorems that we prove that kinematically the instantaneous motion of a rigid body is screw motion.

Formula (6′) for the kinetic energy of a lamina is equally true for a rigid body-provided I' denotes the moment of inertia about the instantaneous axis of rotation. No new proof is needed. The term $M\boldsymbol{u}\cdot\bar{\boldsymbol{v}}'$, which must be added to the sum of the translatory and rotatory energies, will in general not vanish. It will vanish if the point of reference is the center of gravity or any point of the axis of the instantaneous screw (for then $\bar{\boldsymbol{v}}'$ and $\boldsymbol{u}$ are perpendicular). Moreover, if $\boldsymbol{u}\cdot\bar{\boldsymbol{v}}'$ vanishes for any point of the body, it will vanish for all points of the line through that point parallel to the axis of rotation (for $\boldsymbol{u}$ and $\bar{\boldsymbol{v}}'$ do not vary from point to point of this line). Hence from Theorem 5 we may obtain

THEOREM 6. *The kinetic energy of a rigid body in space may be regarded as a sum of the translatory and rotatory energies when and only when the point of reference (fixed in the body) lies on a right circular cylinder passing through the axis of the instantaneous screw and having the center of gravity of the body upon the element diametrically opposite to that axis.*

Angular Momentum. By definition, the angular momenta of each mass and of the whole set of masses about a given fixed point F are respectively

$$m_i\boldsymbol{r}_i \times \boldsymbol{v}_i \quad \text{and} \quad \text{A. M.} = \sum_i m_i\boldsymbol{r}_i \times \boldsymbol{v}_i, \tag{7}$$

where $\boldsymbol{r}$ is the radius from the point F and $\boldsymbol{r} \times \boldsymbol{v}$ denotes the vector product of $\boldsymbol{r}$ into $\boldsymbol{v}$.[1] The angular momentum is therefore a vector magnitude, since $\boldsymbol{r}_i \times \boldsymbol{v}_i$ is a vector perpendicular to the plane of $\boldsymbol{r}_i$ and $\boldsymbol{v}_i$; but, as the direction of the vector is constant in plane motion, the angular momentum in the simple case of plane motion may be treated as a scalar quantity.

[1] The vector product is used to render the product of sums of vectors amenable to the distributive laws and to avoid operating with sines.

We may substitute from (1) in (7) either for $\boldsymbol{r}_i$ or for $\boldsymbol{v}_i$ or for both. The substitution for $\boldsymbol{v}_i$ alone does not seem interesting. The substitution for $\boldsymbol{r}_i$ alone gives

$$\text{A. M.} = \sum_i m_i(\boldsymbol{s} + \boldsymbol{r}_i') \times \boldsymbol{v}_i = M\boldsymbol{s} \times \bar{\boldsymbol{v}} + \sum_i m_i\boldsymbol{r}_i' \times \boldsymbol{v}_i. \tag{8}$$

The interpretation is that the angular momentum about a point F may be computed as the sum of the angular momentum about F of the whole linear momentum $M\bar{\boldsymbol{v}}$ considered as located at O and of the angular momentum about O, but not relative to O (for the actual velocity $\boldsymbol{v}_i$ remains).

The substitution for both $\boldsymbol{r}$ and $\boldsymbol{v}$ gives

$$\begin{aligned}\text{A. M.} &= \sum_i m_i(\boldsymbol{s} + \boldsymbol{r}_i') \times (\boldsymbol{u} + \boldsymbol{v}_i') \\ &= M\boldsymbol{s} \times \boldsymbol{u} + \boldsymbol{s} \times (M\bar{\boldsymbol{v}}') + M\bar{\boldsymbol{r}}' \times \boldsymbol{u} + \sum_i m_i\boldsymbol{r}_i' \times \boldsymbol{v}_i'. \end{aligned} \tag{9}$$

Now $\boldsymbol{s} \times (M\boldsymbol{u})$ is the angular momentum about F of a mass M moving with O and $\Sigma m_i\boldsymbol{r}_i' \times \boldsymbol{v}_i'$ is the angular momentum about and relative to O. As $\bar{\boldsymbol{r}}'$ is the vector from O to G, $-\bar{\boldsymbol{r}}'$ is the vector from G to O, and $\bar{\boldsymbol{r}}' \times (M\boldsymbol{u})$ is the angular momentum, reversed in sign, of a mass M moving with O about G. Finally, $\boldsymbol{s} \times (M\bar{\boldsymbol{v}}')$ is the angular momentum about F of the relative linear momentum $M\bar{\boldsymbol{v}}'$ located at O. Hence

THEOREM 7. *The angular momentum of a system may be written as the sum of* FOUR *terms: (a) the angular momentum about F of the mass of the system located at any moving point O, (b) the relative angular momentum about O, (c) the reversed angular momentum of the mass of the system located at O about, but not relative to, the center of gravity, (d) the angular momentum about F of the relative momentum considered as located at O.*

If we take $O = G$, we have $\bar{\boldsymbol{r}}' = 0$ and $\bar{\boldsymbol{v}}' = 0$. Hence

THEOREM 8. *The angular momentum of a system about a fixed point F is the sum of the angular momentum of the mass of the system moving with the center of gravity and of the angular momentum about and relative to the center of gravity.*

Thus the use of the center of gravity serves to separate the translational and rotational parts of the motion for *both* angular momentum and kinetic energy.

To compute the angular momentum of a rigid lamina we *do not need* Theorem 7; we may apply the original definition (7) and its modification (8) which relates the angular momentum about two different points. For if the lamina has a motion of pure translation, we have A. M. $= M\bar{\boldsymbol{r}} \times \boldsymbol{v}$, where $\bar{\boldsymbol{r}}$ gives the position of the center of gravity and $\boldsymbol{v}$ is the velocity of any point. And if the lamina is rotating, the angular momentum about the instantaneous center is $I_C\omega\boldsymbol{n}$, where I_C is the moment of inertia about the center and $\boldsymbol{n}$ is a unit vector perpendicular to the plane; and the angular momentum about any point F is the sum of $I_C\omega\boldsymbol{n}$ and the angular momentum about F of the linear momentum $M\bar{\boldsymbol{v}}$ located at C.

In many cases, however, it is *convenient* to have an expression for the angular momentum about an arbitrary point F in terms of the motion of a point O fixed

in the lamina and of the motion relative to O. This is given by (9), in which some reductions may now be made. The term $\Sigma m_i \boldsymbol{r}_i' \times \boldsymbol{v}_i'$ becomes $I'\omega\boldsymbol{n}$. If θ be the angle from the direction of $\boldsymbol{s}$ to that of $\boldsymbol{u}$, the first term becomes $M\, s\, u \sin\theta\, \boldsymbol{n}$, where s and u are the magnitudes of the corresponding vectors. The relative velocity $\bar{\boldsymbol{v}}'$ has the magnitude $OG\omega$, and a direction perpendicular to OG. If φ be the angle from $\boldsymbol{s}$ to $OG = \bar{\boldsymbol{r}}'$, the angle from $\boldsymbol{s}$ to $\boldsymbol{v}'$ is $90° + \varphi$, and the term $M\boldsymbol{s} \times \bar{\boldsymbol{v}}'$ reduces to $MsOG\omega \cos\varphi\, \boldsymbol{n}$. If the angle from OG to u be ψ, the term $M\bar{\boldsymbol{r}}' \times \boldsymbol{u}$ is $M\, OG\, u \sin\psi\, \boldsymbol{n}$. Hence

$$\text{A. M.} = M(su \sin\theta + k'^2\omega + OG\omega s \cos\varphi + OG\, u \sin\psi), \tag{9'}$$

where $k'^2 = I'/M$ and where the unit vector $\boldsymbol{n}$ has been thrown out.

By introducing the instantaneous center C, we may write $u = CO\omega$. The angle ψ from OG to $\boldsymbol{u}$ is connected with the angle ψ' from CO to OG by the equation $\psi' + \psi = 90°$. Hence $\sin\psi = \cos\psi'$. Now $CO\cos\psi'$ and $s\cos\varphi = FO\cos\varphi$ are respectively the projections of CO and FO on OG. Hence

$$\text{A. M.} = [M(su \sin\theta + k'^2\omega)] + [M\, OG\, \omega(\text{proj. } FO + \text{proj. } CO)]. \tag{9''}$$

The first bracket contains the terms which we should find for the angular momentum of a lamina if we followed the kinematic resolution of the motion into a translation of any point and a rotation about that point. The second bracket contains the correction terms necessary for the correct dynamic result.

As the sum of the projections is equal to the projection of the sum, we may form the vector $OF + OC = \boldsymbol{R}$, which runs from O to the middle point of FC and as far again. The correction terms then reduce to $-\, M\, OG\, \omega$ proj. $\boldsymbol{R}$, the projection of $\boldsymbol{R}$ being along OG. This will vanish when $\boldsymbol{R}$ is perpendicular to OG, that is, when O lies on a circle with G and the middle point of FC as diametrically opposite points. Hence

THEOREM 9. *The kinematic resolution of the motion gives the correct result for the angular momentum about a point F when and only when the chosen point O, fixed in the lamina, lies on a circle, passing through the center of gravity G and the middle point Q of the line joining the point F and the instantaneous center C, and having GQ as a diameter.*

THEOREM 10. *If a point of reference O is fixed in the lamina, there are ∞^1 points F about which the angular momentum reduces to the sum of two terms, $M(su \sin\theta + k'^2\omega)$, and the locus of these points is a straight line perpendicular to OG and so placed that the segments CF from the instantaneous center to F are bisected by the perpendicular to OG at O.*

This follows from the preceding theorem.

For motion in space the condition $\boldsymbol{s} \times \bar{\boldsymbol{v}}' + \bar{\boldsymbol{r}}' \times \boldsymbol{u} = 0$, which reduces (9) to two terms, is a vector equation. The equivalent three scalar equations have in general three simultaneous solutions, of which one gives the center of gravity and the other two (which may be real or imaginary) are complicated and apparently without interest. *The kinematic resolution is therefore of general dynamic value in space only when the point of reference is the center of gravity.*

Forces and Moments. If forces $\boldsymbol{F}_i$ act on the masses m_i, we have

$$m_i \frac{d\boldsymbol{v}_i}{dt} = \boldsymbol{F}_i \quad \text{and} \quad \sum_i m_i \frac{d\boldsymbol{v}_i}{dt} = M\frac{d\bar{\boldsymbol{v}}}{dt} = \sum_i \boldsymbol{F}_i. \tag{10}$$

This is the theorem on the motion of the center of gravity. Also

$$\sum_i m_i \boldsymbol{r}_i \times \frac{d\boldsymbol{v}_i}{dt} = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i,$$

Hence

$$\sum_i m_i \boldsymbol{s} \times \frac{d\boldsymbol{v}_i}{dt} + \sum_i m_i \boldsymbol{r}_i' \times \frac{d\boldsymbol{u}}{dt} + \sum_i m_i \boldsymbol{r}_i' \times \frac{d\boldsymbol{v}_i'}{dt} = \sum_i \boldsymbol{s} \times \boldsymbol{F}_i + \sum_i \boldsymbol{r}_i' \times \boldsymbol{F}_i$$

or

$$\boldsymbol{s} \times \left(M\frac{d\bar{\boldsymbol{v}}}{dt} \right) + M\bar{\boldsymbol{r}}' \times \frac{d\boldsymbol{u}}{dt} + \sum_i m_i \boldsymbol{r}_i' \times \frac{d\boldsymbol{v}_i'}{dt} = \boldsymbol{s} \times \sum_i \boldsymbol{F}_i + \sum_i \boldsymbol{r}_i' \times \boldsymbol{F}_i.$$

The first terms on the two sides cancel by (10). Hence

$$M\bar{\boldsymbol{r}}' \times \frac{d\boldsymbol{u}}{dt} + \sum_i m_i \boldsymbol{r}_i' \times \frac{d\boldsymbol{v}_i'}{dt} = \sum_i \boldsymbol{r}_i' \times \boldsymbol{F}_i. \tag{11}$$

But

$$\frac{d}{dt}(\boldsymbol{r}_i' \times \boldsymbol{v}_i') = \frac{d\boldsymbol{r}_i'}{dt} \times \boldsymbol{v}_i' + \boldsymbol{r}_i' \times \frac{d\boldsymbol{v}_i'}{dt} = \boldsymbol{r}_i' \times \frac{d\boldsymbol{v}_i'}{dt}.$$

Hence

$$M\bar{\boldsymbol{r}}' \times \frac{d\boldsymbol{u}}{dt} + \frac{d}{dt}\sum_i m_i \boldsymbol{r}_i' \times \boldsymbol{v}_i' = \sum_i \boldsymbol{r}_i' \times \boldsymbol{F}_i. \tag{12}$$

THEOREM 11. *The moment of the forces about any point O is equal to the rate of change of angular momentum about and relative to O plus the mass of the system times the vector product of the relative coordinate $\bar{\boldsymbol{r}}' = OG$, of the center of gravity, and the acceleration of O.*

This theorem is equally applicable to the plane and to space, to a rigid and to a non-rigid body. If there are internal actions and reactions between the particles of the system in the lines joining the particles, these actions and reactions drop out. The theorem includes Huntington's theorems. We may state as corollaries:

THEOREM 12. *The moment of the (external) forces applied to the system is equal to the rate of change of angular momentum when and only when the point of reference (a) is the center of gravity, (b) is fixed in space, (c) is moving with a uniform velocity, or (d) has a vector acceleration passing through the center of gravity.*

THEOREM 13. *If the system is a rigid lamina and the point of reference is fixed in the body and satisfies the conditions of Theorem 12, then*

$$I'\frac{d\omega}{dt} = \text{moment of forces.} \tag{13}$$

In the case of a rigid body in space the angular momentum is a linear vector function (relative to axes fixed in the body) of the angular velocity regarded as a vector, and this must be considered in forming the equation corresponding to (13).

THE THEOREM OF MOMENTS.

By DR. C. G. KNOTT, University of Edinburgh.

Professor Huntington's recent communication on the "Theorem of Moments" (in the AMERICAN MATHEMATICAL MONTHLY for December, 1914) calls attention to a point of real importance. He designedly confined his discussion to the elementary case of uniplanar rigid dynamics; but it struck me that the extraordinarily simple proof of the general case in terms of quaternion analysis might be of interest to readers of the MONTHLY.

I assume Newton's second and third laws of motion or their equivalents.

Let the mass m in position ρ be acted on by the force β. Then for one particle[1]

$$m\ddot{\rho} = \beta$$

and for a system of particles

$$\Sigma m\ddot{\rho} = \Sigma\beta \tag{1}$$

in which $\Sigma\beta$ includes only the applied or external forces (Lex III).

Also, introducing each corresponding ρ as multiplier, and summing for the system, we have

$$\Sigma m\rho\ddot{\rho} = \Sigma\rho\beta. \tag{2}$$

This breaks up into the scalar and vector parts

$$\Sigma Sm\rho\ddot{\rho} = \Sigma S\rho\beta$$

and

$$\Sigma Vm\rho\ddot{\rho} = \Sigma V\rho\beta. \tag{3}$$

The first of these (2) is the "virial" equation and was first used by Hamilton.

The latter equation (3) expresses the theorem of moments; and in it also, by virtue of Lex III, $\Sigma V\rho\beta$ includes only the applied forces.

Now let π be the vector position of any point at rest or in motion, and let σ be the vector position with respect to this new point of any element of the system. Then

$$\rho = \pi + \sigma, \qquad \ddot{\rho} = \ddot{\pi} + \ddot{\sigma}.$$

Equations (1) and (2) become

$$\Sigma m(\ddot{\pi} + \ddot{\sigma}) = \Sigma\beta, \tag{1'}$$

$$\Sigma m(\pi + \sigma)(\ddot{\pi} + \ddot{\sigma}) = \Sigma(\pi + \sigma)\beta. \tag{2'}$$

Since π is the vector position of a definite point, it may be taken outside or placed within the summation symbol at will. Multiplying π into (1′) and subtracting from (2′) we find

$$\Sigma m\sigma(\ddot{\pi} + \ddot{\sigma}) = \Sigma\sigma\beta$$

[1] See my edition (1904) of Kelland & Tait's *Introduction to Quaternions*, page 160.

and the vector part of this is

$$V(\Sigma m\sigma)\ddot{\pi} + \Sigma V m\sigma\ddot{\sigma} = \Sigma V\sigma\beta. \tag{4}$$

Now if the point π is to be a point with regard to which we may apply the theorem of moments in the form (3), we must have

$$\Sigma V m\sigma\ddot{\sigma} = \Sigma V\sigma\beta \tag{5}$$

and hence the condition to be satisfied by π is simply

$$V(\Sigma m\sigma)\ddot{\pi} = 0. \tag{6}$$

But, in general, $\Sigma m\sigma = \mu\Sigma m$, where μ is the vector position of the center of mass of the system referred to the point π. Hence (6) becomes

$$V\mu\ddot{\pi} = 0. \tag{7}$$

This equation may be satisfied by any one of the conditions

(1) $\mu = 0$; (2) $\ddot{\pi} = 0$; (3) $\ddot{\pi}$ parallel to μ.

Hence we may legitimately use the point whose vector is π as a pivot for moments when

(1) it is the center of mass; or

(2) it has no acceleration so that its velocity is constant (the case of) rest or uniform motion in a straight line; or

(3) its acceleration is directed towards the center of mass.

Since the acceleration of the center of mass is necessarily directed through its own position, condition (3) really includes (1). And thus, quite generally, the theorem of moments may be applied with reference to any point whose acceleration is either zero or directed through the center of mass.

BOOK REVIEWS

MATHEMATICS IN DR. ELIOT'S FIVE-FOOT SHELF OF BOOKS.

By W. H. BUSSEY, University of Minnesota.

Several years ago the attention of the whole country was attracted by the announcement made by Dr. Charles W. Eliot that he proposed to select a group of books, filling a five-foot shelf, in the belief that "the faithful and considerate reading of these books, with such re-readings and memorizings as individual taste may prescribe, will give any man the essentials of a liberal education, even if he can devote to them but fifteen minutes a day." The selection was made by him in due time, and the books have since been published by P. F. Collier and Son in a fifty volume set called "The Harvard Classics." In the introduction

to the set, Dr. Eliot states his purpose as follows: "My purpose in selecting The Harvard Classics was to provide the literary materials from which a careful and persistent reader might gain a fair view of the progress of man, observing, recording, inventing, and imagining, from the earliest historical times to the close of the nineteenth century."

The purpose of this review is not to discuss the merits of this five-foot shelf of books, but rather to call to the attention of the readers of the MONTHLY those parts of the set which are of special interest to mathematicians. The set is supposed to furnish a careful reader with the essentials of a liberal education, and surely no such thing is possible without some acquaintance with mathematics and the related sciences, physics and astronomy. Dr. Eliot says in the introduction that he "proposed to make such a selection as any intellectually ambitious American family might use to advantage, even if their early opportunities of education had been scanty."

In such a set of books one would not expect to find any technical mathematical works like Euclid's "Elements" or Newton's "Principia," but one would expect to find something about the great realm of thought which has been explored by means of mathematics.

Volume 30, entitled "Scientific Papers," contains an interesting essay by Simon Newcomb on "The Extent of the Universe." It is preceded by a short introductory note telling something of Newcomb's life and work. The essay itself and the following quotation from the introductory note give some idea of how the mathematical sciences are represented in The Harvard Classics.

"Newcomb's chief labors were in the department of mathematical astronomy, and were directed toward the explanation of the observed movements of the heavenly bodies. The difficulty and complexity of the calculations involved are beyond the conception of the layman; and the achievements which brought Newcomb honors from the learned of almost all civilized countries have to be taken on trust by the general public. He had, nevertheless, an admirable power of clear exposition of those parts of his subject which were capable of popularization; and the accompanying paper is a good example of the simple treatment of a large subject."

More information about the editor's intention concerning scientific work is gained from the following quotations from the introduction.

"Nothing has been included in the series which does not possess good literary form." "It was hard to make up an adequate representation of the scientific thought of the nineteenth century, because much of the most productive scientific thought has not yet been given a literary form. The discoverers' original papers on chemistry, physics, geology, and biology have usually been presented to some scientific society, and have naturally been expressed in technical language, or have been filled with details indispensable from the scientific point of view but not instructive for the public in general."

There is a general index of more than 300 pages in volume 50, and the writer of this review naturally looked to see what he could find under the index word *mathematics*. The result was disappointing, not because the word was missing but because many of the references were trivial. For example, the reference to "Franklin on mathematicians" was to the following paragraph of his "Autobiography" which describes one of the members of a club organized by Franklin.

"Thomas Godfrey, a self-taught mathematician, great in his way, and afterward inventor of what is now called Hadley's Quadrant. But he knew little out of his way, and was not a pleasing

companion; as, like most great mathematicians I have met with, he expected universal precision in everything said, or was forever denying or distinguishing upon trifles, to the disturbance of all conversation."

Having found several references just as trivial, the writer decided that either the set of books did not contain much of special interest to mathematicians or the maker of the index had failed to find such material. The writer then looked through the set carefully for that which he had in mind to find, and discovered the following things of interest.

Volume 30 has already been referred to as containing an essay by Simon Newcomb, who was president of the American Mathematical Society for the two years 1897–98. Every American mathematician knows something of his work and ought to be interested in this paper on "The Extent of the Universe" (Vol. 30, pages 323–336). It contains no mathematics except some ordinary arithmetic and a fact or two from solid geometry. In this same volume, which is entitled "Scientific Papers," are the following papers of interest to mathematicians: Six lectures by Faraday on "The Forces of Matter" (Vol. 30, pages 1–88), especially lectures 1 and 2 on gravitation; a lecture by Helmholtz on "The Conservation of Force" (Vol. 30, pages 180–220); and two lectures by Lord Kelvin on "The Wave Theory of Light" (Vol. 30, pages 263–286), and "The Tides" (Vol. 30, pages 287–321).

Volume 34, entitled "French and English Philosophers," contains a number of interesting things. First of all there is DesCartes's well-known "Discourse on Method" (Vol. 34, pages 5–62) which is preceded by a short biography of DesCartes. Mathematicians will be especially interested in Part 1 in which are found various considerations touching the sciences.

Next come Voltaire's "Lettres Philosophiques" which are also known as "Letters on the English" (Vol. 34, pages 65–162). They represent Voltaire's scientific and philosophic interests. Letter 12 is a letter on Lord Bacon. It begins with the following eulogy of Sir Isaac Newton.

> "Not long since the trite and frivolous question following was debated in a very polite and learned company, viz., who was the greatest man, Cæsar, Alexander, Tamerlane, Cromwell, etc.? Somebody answered that Sir Isaac Newton excelled them all. The gentleman's assertion was very just; for if true greatness consists in having received from heaven a mighty genius, and in having employed it to enlighten our own mind and that of others, a man like Sir Isaac Newton, whose equal is hardly found in a thousand years, is a truly great man. And those politicians and conquerors (and all ages produce some) were generally so many illustrious wicked men. That man claims our respect who commands over the minds of the rest of the world by the force of truth, not those who enslave their fellow creatures: he who is acquainted with the universe, not they who deface it. Since, therefore, you desire me to give you an account of the famous personages whom England has given birth to, I shall begin with Lord Bacon, Mr. Locke, Sir Isaac Newton, etc. Afterwards the warriors and ministers of state shall come in their order."

The next letter (No. 13) "On Mr. Locke" contains this comment on DesCartes: "Our DesCartes, born to discover the errors of antiquity, and at the same time to substitute his own." Voltaire is here referring to DesCartes' theory of vortices about which he has more to say in the next two letters, No. 14 "On DesCartes and Sir Isaac Newton" and No. 15 "On Attraction." The first of

these two contrasts the lives and rival theories of DesCartes and Newton. It contains considerable biography of both men. Voltaire refers to Newton as follows: "This famous Newton, this destroyer of the Cartesian system, died in March, 1727. His countrymen honoured him in his life time, and interred him as though he had been a king who had made his people happy." The latter half of the letter is a defense of DesCartes against those who, in their enthusiasm for Newton, were inclined to belittle him and all his work. Voltaire has this to say in reply to one who presumed to say that DesCartes was not a great geometrician:

"Those who make such a declaration may justly be reproached with flying in their master's face. DesCartes extended the limits of geometry as far beyond the place where he found them as Sir Isaac did after him. The former first taught the method of expressing curves by equations. This geometry which, thanks to him for it, is now grown common, was so abstruse in his time, that not so much as one professor would undertake to explain it; and Schotten in Holland and Fermat in France were the only men who understood it."

The letter "On Attraction" begins as follows:

"The discoveries which gained Sir Isaac Newton so universal a reputation relate to the system of the world, to light, to geometrical infinities, and lastly to chronology, with which he used to amuse himself after the fatigue of his severer studies. I will now acquaint you (without prolixity f possible) with the few things which I have been able to comprehend of all these sublime ideas."

This paragraph is really an introduction to three letters of which the one "On Attraction" is the first. Letter 16 is a popular account of "Sir Isaac Newton's Optics" and Letter 17 is "On Infinities in Geometry and Sir Isaac Newton's Chronology." The phrase "Infinities in Geometry" refers to differential and integral calculus. In the letter "On Attraction" Voltaire tells the well known story of the falling apple starting in Newton's mind the train of thought which led him to his theory of gravitation.

The following quotation from the beginning of Letter 17 tells what kind of impression the infinitesimal calculus made on a layman:

"The labyrinth and abyss of infinity is also a new course Sir Isaac Newton has gone through, and we are obliged to him for the clue by whose assistance we are enabled to trace its various windings. DesCartes got the start of him also in this astonishing invention. He advanced with mighty steps in his geometry, and was arrived at the very borders of infinity, but went no farther. Dr. Wallis, about the middle of the last century, was the first to reduce a fraction by perpetual division to an infinite series. The Lord Brouncker employed this series to square the hyperbola. Mercator published a demonstration of this quadrature; about which time Sir Isaac Newton, being then twenty-three years of age, had invented a general method to perform on all geometrical curves what had just before been tried on the hyperbola. It is to this method of subjecting everywhere infinity to algebraical calculations that the name is given of differential calculations or of fluxions and integral calculation. It is the art of numbering and measuring exactly a thing whose existence cannot be conceived."

Voltaire has this to say about the controversy over the invention of the calculus:

"For many years the invention of this famous calculation was denied to Sir Isaac Newton. In Germany Mr. Leibnitz was considered as the inventor of the differences or moments, called fluxions, and Mr. Bernoulli claimed the integral calculus. However, Sir Isaac is now thought to have first made the discovery, and the other two have the glory of having once made the world doubt whether it was to be ascribed to him or to them."

The six letters of Voltaire's which have been mentioned make 35 pages of very interesting reading. While reading them, one should bear in mind the dates of the three men, DesCartes (1596–1650), Newton (1642–1727), and Voltaire (1694–1778).

Volume 39, entitled "Famous Prefaces," contains the "Dedication of the Revolutions of the Heavenly Bodies" by Nicholaus Copernicus (1543). The book which was introduced by this dedication laid the foundation of modern astronomy. At the time when it was written, the earth was believed by all to be the fixed center of the universe; and although many of the arguments used by Copernicus were invalid and absurd, he was the first modern to put forth the heliocentric theory as "a better explanation." It remained for Kepler, Galileo and Newton to establish the theory on firm grounds. The dedication is to Pope Paul III. Copernicus explains how he came to write the book and the long delay in publishing it. This volume of The Harvard Classics also contains the preface to Newton's "Philosophiæ Naturalis Principia Mathematica." Mathematicians of to-day do not need to read this work of Newton's as a part of their mathematical education because they can get the substance of it from later works, but they will find it worth while to read what Newton had to say about it in his preface.

Volume 48 is devoted entirely to the works of Pascal, but of course it is Pascal the philosopher rather than Pascal the mathematician. It contains his "Thoughts" and some of his letters and minor works. Section 1 of the "Thoughts" is entitled "Thoughts on mind and style." Some of these refer to mathematics. In fact Thought No. 1 is on "The difference between the mathematical and the intuitive mind." Section 6 is on "The Philosophers." Here is a quotation from this section, being Thought No. 340. "The arithmetical machine produces effects which approach nearer to thought than all the actions of animals. But it does nothing which would enable us to attribute will to it, as to the animals." Later in the volume there is a letter from Pascal to Queen Christina of Sweden on sending her the arithmetical machine. The following quotation is from Thought No. 908 of Section 14 entitled "Polemical Fragments." "But is it probable that probability gives assurance?" Of course Pascal was not thinking of the mathematical theory of probability when he wrote this epigram, for that branch of mathematics was only in its infancy in his day; but we may well ask the question with reference to some of the conclusions which have been drawn from the theory of probability.

Of Pascal's minor works included in this volume, the only one of special interest to mathematicians is an essay "On the Geometrical Spirit." It consists of an introduction on page 427; Section I which is "On the method of geometrical, that is, of methodical and perfect demonstrations" on pages 428–444; and Section II on "The Art of Persuasion," which the editor has, for some reason, printed as a separate work on pages 406–417. In addition to what the title would lead one to expect, this work "On the Geometrical Spirit" contains a long argument against those who believed in "indivisibles." At the end of his argument Pascal says that "those who will not be satisfied with these reasons, and will

persist in the belief that space is not divisible *ad infinitum*, can make no pretensions to geometrical demonstrations, and although they may be enlightened in other things, they will be very little in this; for one can easily be a very capable man and a bad geometrician." In the course of his argument, Pascal has some interesting things to say about the relations of unity and zero to the number system. Near the beginning of Section I, Pascal has considerable to say about definitions and the necessity of leaving some things in geometry undefined. He has this to say of those who insist on defining everything:

"For there is nothing more feeble than the discourse of those who wish to define these primitive words (space, time, motion, number, equality). What necessity is there, for example, of explaining what is understood by the word *man*? Do we not know well enough what the thing is that we wish to designate by the term? And what advantage did Plato think to procure us by saying that he was a two-legged animal without feathers? As though the idea that I have of him naturally, and which I cannot express, were not clearer and surer than that which he gives me by his useless and even ridiculous explanation; since a man does not lose humanity by losing the two legs, nor does a capon acquire it by losing his feathers."

Volume 48 closes with Pascal's "Preface to the Treatise on Vacuum" and a very short "New Fragment of the Treatise on Vacuum." These have reference to Pascal's experimental work on atmospheric pressure. He repeated the experiments of Torricelli by which the pressure of the atmosphere could be estimated as a weight; and by what has been called "The Great Experiment of the Puy-de-Dôme" he confirmed Torricelli's theory of the cause of barometrical variations. In this experiment Pascal obtained at the same instant readings of the height of a column of mercury in a barometer tube at different altitudes.

The parts of The Harvard Classics which have been mentioned as representing the mathematical sciences are all contained in the four volumes 30, 34 39, 48. The number of pages in these parts is more than 250. The authors represented are Newcomb, Faraday, Kelvin, Helmholtz, DesCartes, Voltaire, Copernicus, Newton, and Pascal. The parts written by Voltaire tell at some length of the work of DesCartes and Newton, and contain references to Bernoulli, Leibnitz, Wallis, Fermat and Mercator. Thus it is seen that the mathematical sciences have not been altogether neglected. An interesting question for readers of the MONTHLY is this: What else relating to the mathematical sciences might Dr. Eliot have included in his "Five-Foot Shelf of Books"?

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

PROBLEMS FOR SOLUTION.

ALGEBRA.

When this issue was made up, no solutions had been received for number 406.

436. Proposed by WALTER H. DRANE, Cumberland University, Tenn.

The product of two numbers p and q may be obtained by dividing p by 2 successively, discarding remainders, until the quotient 1 is obtained and then multiplying q by 2 successively and

adding those products, $2^i q$, which correspond to those quotients of p that are odd numbers. E. g. 51×81. 25, 162; 12, 324; 6, 648; 3, 1,296; 1, 2,592.

$$51 \times 81 = 81 + 162 + 1{,}296 + 2{,}592 = 4{,}131.$$

Prove or disprove the truth of the above proposition.

437. Proposed by C. N. SCHMALL, New York City.

Given that $S_1, S_2, S_3, \cdots, S_k$ are the sums of k arithmetical series, each taken to n terms. The first terms are respectively $1, 2, 3, \cdots, k$, and the common differences are $1, 3, 5, \cdots, (2k-1)$. Show that

$$S_1 + S_2 + S_3 + \cdots + S_k = \frac{nk(nk+1)}{2}.$$

GEOMETRY.

When this issue was made up, no solutions had been received for numbers 446 and 449.

466. Proposed by HORACE OLSON, Chicago, Illinois.

Given the edges of a triangular pyramid, find the radius of the inscribed sphere.

467. Proposed by E. T. BELL, Seattle, Washington.

It is well-known that if i, j, k, l are concyclic points, W_i the Wallace line (frequently, and erroneously, called the Simson line), of i with respect to the triangle jkl, then W_i, W_j, W_k, W_l are concurrent, say in the point $\{i, j, k, l\}$. If $1, 2, 3, \cdots$ denote concyclic points, prove that:

(i) $\{1, 2, 3, 4\}$, $\{1, 2, 3, 5\}$, $\{1, 2, 4, 5\}$, $\{1, 3, 4, 5\}$, $\{2, 3, 4, 5\}$ are concyclic; say on the circle $[1, 2, 3, 4, 5]$;

(ii) Starting with 1, 2, 3, 4, 5, 6, omitting each point in turn, by (i), six circles, are found; these are concurrent, say in the point $\{1, 2, 3, 4, 5, 6\}$;

(iii) Starting with 1, 2, 3, 4, 5, 6, 7, seven points of the kind in (ii) are found; these lie on a circle.

(iv) Continuing thus indefinitely, there is, in each case, finally a unique point or circle according as the number of initial points is even or odd. Also, at any stage, the point of concurrence or the circle bears a simple relation to the initial points: what is it?

[*Note.*—This problem is obviously connected with the theorems of Clifford (*Mathematical Papers*, pp. 38 and 410), but it is interesting to note that it may be completely solved by the methods of Euclid, Books I to III (but at considerable length), and hence is well within the range of high school students. But a very simple proof may be given by the methods of strictly elementary analytical geometry.]

468. Proposed by ELMER SCHUYLER, Brooklyn, New York.

Given two circles and a straight line, to draw a circle tangent to the line and coaxial with the two given circles.

469. Proposed by J. ALEXANDER CLARKE, West Philadelphia High School.

If in an isosceles triangle, a circle is described on one side as diameter, and a line is drawn through the mid-point of the side parallel to the base, the circle and the parallel will intercept on the *tri*sector of the angle at the vertex a segment equal to the radius of the circle. Show how this can be used to trisect any angle.

CALCULUS.

When this issue was made up, no solutions had been received for numbers 336, 338–340, 342, 348, 353, 360, and 363.

387. Proposed by C. N. SCHMALL, New York City.

Show that the volume bounded by the cone $x^2 + y^2 = (a - z)^2$ and the planes $x = 0$, $x = z$, is $\frac{2}{9}a^3$.

388. Proposed by PAUL CAPRON, U. S. Naval Academy.

If $f(x, y) = 0$ represents (in rectangular coordinates) a curve having a simple tangency to the axis of x at the origin, the value of $x^2/2y$, derived from $f(x, y) = 0$, and evaluated for $x = 0$, $y = 0$, will be the radius of curvature at the origin; or if the curve is similarly tangent to the y-axis at the origin, $y^2/2x$, evaluated for $x = 0$, $y = 0$, is the radius of curvature at the origin.

389. Proposed by FRANK R. MORRIS, Glendale, Calif.

A man is at the southeast corner of a section of land and wishes to walk to the opposite corner in the least possible time. A circular track with a radius of $1/\pi$ miles is located in the section tangent to the west line at a point 120 rods from the south line. Conditions are such that he can walk at the rate of 4 miles an hour inside the track and 3 miles an hour outside the track. What course should he choose and how long is it?

MECHANICS.

When this issue was made up, no solutions had been received for numbers 274, 277, 279, 287, 291, and 292.

311. Proposed by B. J. BROWN, Student in Drury College.

A particle oscillates in a straight line about a center of force varying as the distance, and is subject to a retardation $k \times (\text{vel.})^2$. If a, b be two successive elongations, on opposite sides, prove that $(1 + 2ka)e^{-2ka} = (1 - 2kb)e^{2kb}$. What form does the result take if a is infinite? From *Lamb's Dynamics*, p. 299, ex. 14.

312. Proposed by C. N. SCHMALL, New York City.

A ball of elasticity e is projected upward from a point on an inclined plane, so that after its first contact with the plane it rebounds to its starting point. If ϕ be the inclination of the plane to the horizontal, and ψ the angle made by the line of projection with the inclined plane, show that

$$\cot\phi \cot\psi = e + 1.$$

313. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

A heavy extensible wire of length c and of constant cross-section w, and density k, is suspended by one end and hangs vertically. If e is the coefficient of elasticity, show that the length of the wire when stretched will be $c(1 + ekgw/2)$.

NUMBER THEORY.

When this issue was made up, no solutions had been received for numbers 188–9, 191–2, 196, 200, 205, 208–9, 211, 214–15, 217, and 219.

232. Proposed by E. T. BELL, Seattle, Washington.

If $F(x)$ is any function of x which vanishes with x, and which, for $0 < |x| \gtreqless |\xi|$, can be expanded in an absolutely convergent series of positive powers of x, show that a function $f(n)$ may be found, essentially in one way only, such that

$$\int_0^{\xi} \frac{1}{x} F(x)dx = -\log \prod_{n=1}^{\infty} (1 - \xi^n)^{(1/n)f(n)},$$

and find the form of $f(n)$ explicitly in terms of the coefficients in the expansion of $F(x)$. Hence, as particular examples, expand (when possible by this method) e^x as an infinite product, and show that

$$\frac{1}{e} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{2^n}\right)^{(1/n)\phi(n)}$$

where $\varphi(n)$ is the totient of n.

233. Proposed by V. M. SPUNAR, Chicago, Illinois.

Solve in rational numbers $x^2 + y^2 = a^2$, $xy = m/n$, where m and n are integers and relatively prime to each other.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

424. Proposed by S. A. JOFFE, New York City.

Sum the series

$$\binom{n}{a} - \binom{n-1}{a}\binom{i}{1} + \binom{n-2}{a}\binom{i}{2} - \binom{n-3}{a}\binom{i}{3} + \cdots + (-1)^i\binom{n-i}{a}$$

and consider the cases $i = a$ and $i > a$.

I. SOLUTION BY FRANK IRWIN, University of California.

The sum is $\binom{n-i}{a-i}$. This will hold for all (positive) values of n, a, i, provided we put $\binom{k}{l} = 0$ if $k < l$ or if l is negative. Thus for $i > a$, the sum will be zero; while for $n - i < a$, some of the terms of the sum will drop out.

First Proof.—These statements may be proved by mathematical induction. For they are true when $i = 1$; suppose them then to be true for some particular value of i and all values of n. We shall show that they are true for $i + 1$. For we have:

$$\binom{n}{a} - \binom{n-1}{a}\binom{i}{1} + \cdots \pm \binom{n-k}{a}\binom{i}{k} + \cdots = \binom{n-i}{a-i},$$

$$\binom{n-1}{a} - \binom{n-2}{a}\binom{i}{1} + \cdots \mp \binom{n-k}{a}\binom{i}{k-1} + \cdots = \binom{n-i-1}{a-i};$$

whence, subtracting, we obtain

$$\binom{n}{a} - \binom{n-1}{a}\binom{i+1}{1} + \cdots \pm \binom{n-k}{a}\left[\binom{i}{k} + \binom{i}{k-1}\right] + \cdots$$
$$= \binom{n-i}{a-i} - \binom{n-i-1}{a-i},$$

or

$$\binom{n}{a} - \binom{n-1}{a}\binom{i+1}{1} + \cdots \pm \binom{n-k}{a}\binom{i+1}{k} + \cdots = \binom{n-i-1}{a-i-1},$$

the desired result.

Second Proof.—$\binom{n-i}{a-i}$ gives the combinations of $n - i$ objects, say white balls, $a - i$ at a time. We may count these combinations in another way. Let us add to the white balls i black balls, and consider the $\binom{n}{a}$ combinations of black and white balls together, a at a time. If we take one of these combinations *that contains all the black balls* and discard the latter, we get one of the combinations of the white balls $a - i$ at a time; and each of these latter combinations may be obtained in this way. To count these latter, then, we must subtract from the $\binom{n}{a}$ combinations above those that omit any of the black balls. There are $\binom{n-1}{a}$ that omit a particular black ball, $\binom{i}{1}$ black balls; but the number

$$\binom{n}{a} - \binom{n-1}{a}\binom{i}{1}$$

is too small, since a combination that omits more than one black ball has been subtracted, in the second term, more than once. It is easily seen how, proceeding along these lines, we obtain our given sum

$$\binom{n}{a} - \binom{n-1}{a}\binom{i}{1} + \binom{n-2}{a}\binom{i}{2} - \cdots,$$

and it remains to examine how often herein a particular combination, say one that omits k black balls, has been reckoned. A little consideration will show that it has been counted $1 - \binom{k}{1} + \binom{k}{2} - \cdots \pm 1 = 0$ times; so that the only combinations counted are those that contain all i black balls, and our sum is equal to the number of these combinations, that is, to $\binom{n-i}{a-i}$, as was to be proved.

When $i > a$, $a - i$ is negative and therefore $\binom{n-i}{a-i} = 0$.

II. Solution by the Proposer.

By an elementary theorem in the calculus of finite differences, we have

$$\Delta^i u_x = u_{x+i} - \binom{i}{1} u_{x+i-1} + \binom{i}{2} u_{x+i-2} - \cdots + (-1)^i u_x.$$

Making here $x + i = n$, or $x = n - i$, we obtain

$$u_n - \binom{i}{1} u_{n-1} + \binom{i}{2} u_{n-2} - \cdots + (-1)^i u_{n-i} = \Delta^i u_{n-i}.$$

If we take

$$u_n = \binom{n}{a}, \text{ so that } u_{n-1} = \binom{n-1}{a}, \ u_{n-2} = \binom{n-2}{a}, \text{ etc.,}$$

the last equation becomes

$$\binom{n}{a} - \binom{i}{1}\binom{n-1}{a} + \binom{i}{2}\binom{n-2}{a} - \cdots + (-1)^i \binom{n-i}{a} = \Delta^i \binom{n-i}{a},$$

the first member of which is the given series.

In order to evaluate the second member, we notice that

$$\Delta \binom{x}{a} = \binom{x+1}{a} - \binom{x}{a}.$$

But as

$$\binom{x+1}{a} - \binom{x}{a} = \frac{(x+1)!}{(x+1-a)!a!} - \frac{x!}{(x-a)!a!} = \frac{x!}{(x+1-a)!(a-1)!} = \binom{x}{a-1},$$

therefore

$$\Delta \binom{x}{a} = \binom{x}{a-1}, \quad \Delta^2 \binom{x}{a} = \Delta \binom{x}{a-1} = \binom{x}{a-2}, \quad \Delta^3 \binom{x}{a} = \binom{x}{a-3}, \quad \text{etc.}$$

and in general

$$\Delta^i \binom{x}{a} = \binom{x}{a-i}; \quad \Delta^i \binom{n-i}{a} = \binom{n-i}{a-i}.$$

Hence

$$\binom{n}{a}-\binom{i}{1}\binom{n-1}{a}+\binom{i}{2}\binom{n-2}{a}-\cdots+(-1)^i\binom{n-i}{a}=\binom{n-i}{a-i}. \qquad (1)$$

In the case when $i = a$, we have $a - i = 0$, and $\binom{n-i}{a-i} = 1$; therefore, equation (1) becomes

$$\binom{n}{a}-\binom{a}{1}\binom{n-1}{a}+\binom{a}{2}\binom{n-2}{a}-\cdots+(-1)^a\binom{n-a}{a}=1. \qquad (2)$$

When $i > a$, then $a - i$ is negative, so that $\binom{n-i}{a-i} = 0$. Hence for $i > a$,

$$\binom{n}{a}-\binom{i}{1}\binom{n-1}{a}+\binom{i}{2}\binom{n-2}{a}-\cdots+(-1)^i\binom{n-i}{a}=0. \qquad (3)$$

An excellent solution by a somewhat different method from the ones exhibited in these two solutions was received from A. M. Harding.

425. Proposed by CLIFFORD N. MILLS, Brookings, S. D.

Solve for x and y the equations $2^{x+y} = 6$, $2^{x+1} = 3^y$.

Solution by Elijah Swift, University of Vermont.

We shall agree to mean by a^x, where a is real, $e^{x \log a}$, in which the real logarithm of a is to be taken, so that a^x is single valued. We have, then, from the given equations, $(x + y) \log 2 = \log 6$, $(x + 1) \log 2 = y \log 3$, where all the logarithms are real. Solving these linear equations for x and y, we obtain the required solutions

$$x = \frac{(\log 6)^2 - \log 2 \cdot \log 12}{\log 2 \log 6} = 1.198 \text{ and } y = \frac{\log 12}{\log 2} = 1.387.$$

On account of the homogeneity of these fractions in the logarithmic function, we may substitute logarithms to the base 10 in our computation of the numerical values of x and y.

Also solved by Emma Gibson, Nathan Altshiller, A. L. McCarty, A. M. Harding, Horace Olson, Richard Morris, G. W. Hartwell, C. E. Githens, Frank Irvin, Elizabeth B. Davis, F. L. Carmichael, Edward S. Ingham, S. A. Joffe, W. C. Eells, Elmer Schuyler, V. M. Spunar, R. M. Mathews, and Cyril A. Nelson.

GEOMETRY.

454. Proposed by LOUIS ROUILLION, Mechanics Institute, New York City.

Show how to construct an equilateral triangle with its vertices lying on three [parallel] lines not equally spaced.

I. Solution by C. N. Schmall, New York City.

The proposer evidently meant that the three lines are parallel.

(i) Let x, y, and z be the three parallel lines in order. At any point, M, on the line y, construct the angles AMC and BMC each equal to 60° (Fig. 1). Draw the line AB, and, about the triangle AMB, circumscribe the circle ABC.

Draw the lines AC and BC. Then, the triangle ABC is equilateral as is easily seen and, therefore, satisfies the condition of the problem.

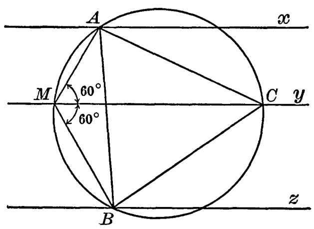

Fig. 1.

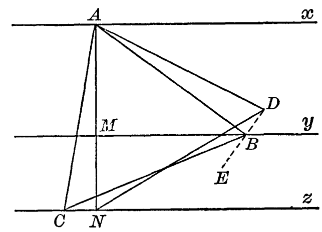

Fig. 2.

(ii) We may also construct the triangle as follows: Through the point, M (Fig. 2), draw AN perpendicular to the line y, and on AN as a side construct the equilateral triangle AND. Through D, draw ED perpendicular to AD, meeting the line y in B. Draw the line AB. Then draw AC, making the angle NAC equal to the angle BAD. Draw BC. Then the triangle ABC is equilateral, and has its vertices on the three given lines as is required. For, the two right triangles BAD and CAN are congruent, having, by construction, a side and an angle of one made equal to a side and an angle of the other. Hence, side AC equals side AB. Angle CAB equals angle $NAD = 60°$. Hence, angles ACB and ABC are equal and the triangle is equilateral.

II. Solution by Nathan Altshiller, University of Washington.

Let us generalize the problem in the following way:

Construct a triangle ABC so that its vertices A, B, C shall lie respectively on three given lines p, q, r, while its angles α, β, γ shall have each a given value.

The condition that the triangle shall have three given angles is equivalent to the condition that one of its angles, say α, shall have a given value, and that the ratio AB/AC, of the sides AB and AC including this angle, shall be equal to a given quantity k.

Let D be the foot of the perpendicular AD dropped from A upon the line r. Rotate the triangle ACD bodily about the point A into the position $AC'D'$ such that the line AC shall take the direction of the line AB. The angle DAD' is equal to the angle $CAB = \alpha$.

The perpendicular BE dropped from B upon AD', determines on AD' a point E such that

$$\frac{AE}{AD'} = \frac{AB}{AC'} = \frac{AB}{AC} = k.$$

This leads to the following method of solving the problem:

From any point A on p drop the perpendicular AD upon r and construct the line AD' making with AD an angle equal to the given angle α. On AD' take a

segment $AD' = AD$ and determine on it the point E so that $\frac{AE}{AD'} = k$. The perpendicular EB to AD' at the point E will meet q in a point B which is the second vertex, besides A, of the required triangle. The third vertex C is easily found.

This construction is applicable whether the given lines are concurrent, parallel, or form a triangle.

To satisfy the conditions of the generalized problem, it is necessary to exercise care in effecting the rotation in the proper sense. If the problem should read that ABC is to be similar to a given triangle without specifying which angle shall have its vertex on which of the given lines, the point A would be the origin of three triangles each solving the problem, but differently situated. The three triangles coincide, if the angles of ABC are to be equal to each other, as in the case of the proposed problem.

The particular case was also solved by DANIEL KRETH, ELIZABETH B. DAVIS, CLIFFORD N. MILLS, S. A. JOFFE, ELMER SCHUYLER, J. W. CLAWSON, and the PROPOSER.

455. Proposed by R. P. BAKER, University of Iowa.

Find the minimum triangle of assigned angles inscribed in a given triangle.

SOLUTION BY ROGER A. JOHNSON, Western Reserve University.

By the well-known theorem of MIQUEL (McClelland, *Geometry of the Circle*, pp. 40–42), if a triangle PQR be inscribed in a given triangle ABC so as to be directly similar, i. e., without being turned over, to a given triangle; then the circles AQR, BRP, CPQ pass through a point O, which, moreover, is fixed for all positions of the inscribed triangle; further, if P, Q, R lie on the sides of the triangle, and not on their extensions, $\angle OPB = \angle OQC = \angle ORA$. If from the point O, lines OP_1, OQ_1, OR_1 be drawn, making equal angles with the sides, their extremities P, Q, R, will form a triangle similar to PQR.

It follows that if the point O be known, we shall obtain the minimum triangle by taking the pedal triangle; that is to say, the triangle whose vertices are the feet of the perpendiculars from O to the sides of ABC.

First inscribe in ABC any triangle of the given species. The following, due to Petersen, is, perhaps, the simplest. Take Y on AC and Z on AB, at random. On YZ construct a triangle XYZ of the given type and in the desired position. Let AX cut BC at P. Through P draw PQ and PR parallel to XY and XZ, respectively. Then QR will be parallel to YZ and PQR will be an inscribed triangle of the desired type.

Now draw the circles AQR, BRP, CPQ, intersecting again at O. From O, drop perpendiculars OL, OM, ON, to the sides. Then LMN is the required triangle.

Also solved by C. N. SCHMALL, FRANK IRWIN, and J. W. CLAWSON.

CALCULUS.

368. Proposed by PAUL CAPRON, Annapolis, Maryland.

Develop $\log_{10} x/\sin x$ and $\log_{10} \tan x/x$ to three terms as functions of $\log_{10} \sec x$, showing that

if x is less than $7°\,15'$, then to five decimals, $\log_{10} x = \log_{10} \sin x + \frac{1}{3} \log_{10} \sec x = \log_{10} \tan x - \frac{2}{3} \log_{10} \sec x$.

Solution by the Proposer.

$$y = \log \frac{x}{\sin x}, \quad \text{or} \quad e^y = \frac{x}{\sin x}; \quad \frac{dy}{dx} = \frac{1}{x} - \cot x.$$

$$z = \log \sec x, \quad \text{or} \quad e^z = \sec x; \quad \frac{dz}{dx} = \tan x.$$

Hence,

$$\frac{dy}{dx} = \tan x \frac{dy}{dz}.$$

Also

$$z - y = \log \frac{\tan x}{x}, \quad \text{or} \quad e^{z-y} = \frac{\tan x}{x}.$$

$$\tan x \frac{dy}{dz} = \frac{1}{x} - \cot x; \quad \tan^2 x \frac{dy}{dz} = \frac{\tan x}{x} - 1, \quad \frac{dy}{dz} = \frac{e^{z-y} - 1}{e^{2z} - 1} = \frac{e^{-y} - e^{-z}}{e^z - e^{-z}}.$$

$$z = 0 = y \quad \text{when} \quad x = 0.$$

$$\left.\frac{dy}{dz}\right]_{z=0} = \frac{0}{0} = \left.\frac{-\left.\frac{dy}{dz}\right] e^{-y} + e^{-z}}{e^z + e^{-z}}\right]_{z=0}; \quad \left.\frac{dy}{dz}\right]_{z=0} = \frac{-\left.\frac{dy}{dz}\right]_{z=0} + 1}{2}; \quad \left.\frac{dy}{dz}\right]_{z=0} = \frac{1}{3}.$$

$$\left.\frac{d^2y}{dz^2}\right]_{z=0} = \frac{1}{(e^z + e^{-z})^2}\left[2 - e^{-y}\left([e^z - e^{-z}]\frac{dy}{dz} + [e^z + e^{-z}]\right)\right]; \quad \left.\frac{d^2y}{dz^2}\right]_{z=0} = \frac{0}{0}.$$

$$\left.\frac{d^2y}{dz^2}\right]_{z=0} = \frac{1}{2(e^{2z} - e^{-2z})}\left[-e^{-y}\left[(e^z - e^{-z})\frac{d^2y}{dz^2} - (e^z - e^{-z})\left(\frac{dy}{dz}\right)^2 + (e^z - e^{-z})\right]\right]_{z=0}$$

$$= \frac{-e^{-y}}{2(e^z + e^{-z})}\left[\frac{d^2y}{dz^2} - \left(\frac{dy}{dz}\right)^2 + 1\right]_{z=0}; \quad \left.\frac{d^2y}{dz^2}\right]_{z=0} = -\frac{8}{45}.$$

$$y = \frac{z}{3} - \frac{8}{45}z^2 \cdots, \quad z - y = \frac{2}{3}z + \frac{8}{45}z^2 \cdots,$$

or

$$\log x - \log \sin x = \frac{1}{3} \log \sec x - \frac{8}{45} (\log \sec x)^2,$$

$$\log_{10} x - \log_{10} \sin x = \frac{1}{3} \log \sec x - \frac{8}{45M} (\log \sec x)^2.$$

Similarly,

$$\log_{10} \tan x - \log_{10} x = \frac{2}{3} \log \sec x + \frac{8}{45M} (\log \sec x)^2,$$

$$\frac{8}{45M} (\log_{10} \sec x)^2 = .000005 \text{ if } \log_{10} \sec x = .0034948, \; x = 7°\,15',$$

$$\frac{8}{45M} = .4 \text{ nearly}, \quad = (.40926), \quad = \frac{70}{171} \text{ very closely}.$$

This theorem enables one to dispense without inconvenience with the usual table for small angles. It is well to take $\log_{10}$ c.m. $1' = 6.46372\frac{2}{3} - 10$ in order to account for odd thirds. The theorem gives correct results in most instances for values of θ up to about $10°$. For $\theta = 10°$ there is an error of one unit in the fifth place of decimals.

369. Proposed by I. A. Barnett, Chicago, Ill.

Compute the definite integral $\int_a^b \log x dx$ by direct summation.

I. Solution by A. M. Harding, University of Arkansas.

Let

$$S_n = dx \cdot \log a + dx \cdot \log (a + dx) + dx \cdot \log (a + 2dx) + \cdots + dx \cdot \log [a + (n - 1)dx].$$

Then

$$I = \int_a^b \log x dx = \lim_{n=\infty} S_n, \text{ where } ndx = b - a.$$

Now

$$\log (a + dx) = \log a + \frac{1}{a} dx - \frac{1}{2a^2} dx^2 + \frac{1}{3a^3} dx^3 - \frac{1}{4a^4} dx^4 + \cdots$$

$$\log (a + 2dx) = \log a + \frac{1}{a} 2dx - \frac{1}{2a^2} 2^2dx^2 + \frac{1}{3a^3} 2^3dx^3 - \frac{1}{4a^4} 2^4dx^4 + \cdots$$

$$\log [a + (n - 1)dx] = \log a + \frac{1}{a} (n-1)dx - \frac{1}{2a^2} (n-1)^2dx^2 + \frac{1}{3a^3} (n-1)^3dx^3 - \frac{1}{4a^4} (n-1)^4dx^4 + \cdots.$$

Hence,

$$S_n = \log a \cdot ndx + \frac{1}{a} [1 + 2 + 3 + 4 + \cdots + (n - 1)]dx^2 - \frac{1}{2a^2} [1^2 + 2^2 + 3^2 + \cdots + (n - 1)^2]dx^3$$
$$+ \frac{1}{3a^3} [1^3 + 2^3 + 3^3 + \cdots + (n - 1)^3]dx^4 - \frac{1}{4a^4} [1^4 + 2^4 + 3^4 + \cdots + (n - 1)^4]dx^5 + \cdots$$
$$= (b - a) \log a + \frac{1}{a} (n^2/2 - n/2)dx^2 - \frac{1}{2a^2} (n^3/3 - n^2/2 + n/6)dx^3$$
$$+ \frac{1}{3a^3} (n^4/4 - n^3/2 + n^2/4)dx^4 - \frac{1}{4a^4} (n^5/5 - n^4/2 + n^3/3 - n/30)dx^5$$
$$+ \frac{1}{5a^5} (n^6/6 - n^5/2 + 5n^4/12 - n^2/12)dx^6 - \cdots;$$

$$\therefore I = \lim S_n = (b - a) \log a + \frac{(b - a)^2}{2a} - \frac{(b - a)^3}{2 \cdot 3a^2} + \frac{(b - a)^4}{3 \cdot 4a^3} - \frac{(b - a)^5}{4 \cdot 5a^4} + \frac{(b - a)^6}{5 \cdot 6a^5} - \cdots$$
$$= (b - a) \log a + \left(1 - \frac{1}{2}\right) \frac{(b - a)^2}{a} - \left(\frac{1}{2} - \frac{1}{3}\right) \frac{(b - a)^3}{a^2} + \left(\frac{1}{3} - \frac{1}{4}\right) \frac{(b - a)^4}{a^3}$$
$$+ \left(\frac{1}{4} - \frac{1}{5}\right) \frac{(b - a)^5}{a^4} - \cdots$$
$$= (b - a) \log a + (b - a)[- (1 - b/a) - 1/2(1 - b/a)^2 - 1/3(1 - b/a)^3 - 1/4(1 - b/a)^4 - \cdots]$$
$$+ a\left[-\frac{1}{2}\left(\frac{b - a}{a}\right)^2 + \frac{1}{3}\left(\frac{b - a}{a}\right)^3 - \frac{1}{4}\left(\frac{b - a}{a}\right)^4 + \frac{1}{5}\left(\frac{b - a}{a}\right)^5 - \cdots \right]$$
$$= (b - a) \log a + (b - a) \log \left[1 - \left(1 - \frac{b}{a}\right) \right] + a \left[\log \left\{ 1 - \left(1 - \frac{b}{a}\right) \right\} + \frac{b - a}{a} \right]$$
$$= (b - a) \log a + (b - a) \log \frac{b}{a} + a \log \frac{b}{a} + b - a$$
$$= (b - a) \log a + b \log \frac{b}{a} + b - a$$
$$= b \log b - a \log a + b - a.$$

II. Solution by the Proposer.

Using the method of division indicated in the diagram, we have,

$$q^n = b \qquad \text{and} \qquad n = \log \frac{b}{a} - \log q.$$

Hence,

$$\int_a^b \log x dx = \lim_{q \doteq 1} [a(q-1)\log a + aq(q-1)\log aq + \cdots + aq^{n-1}(q-1)\log aq^{n-1}],$$

$$= \lim_{q \doteq 1} a(q-1)[(1+q+\cdots+q^{n-1})\log a + (q+2q^2+\cdots+(n-1)q^{n-1})\log q],$$

$$= \lim_{q \doteq 1} a(q-1)\left[\left(\frac{q^n-1}{q-1}\right)\log a + \left\{\frac{q-nq^n+(n-1)q^{n+1}}{(1-q)^2}\right\}\log q\right],*$$

$$= \lim_{q \doteq 1} a\left[(q^n-1)\log a + \frac{q(1-q^n)}{q-1}\log q + nq^n \log q\right],$$

$$= \lim_{q \doteq 1} a\left[\left(\frac{b-a}{a}\right)\log a + \frac{q}{q-1}\left(\frac{a-b}{a}\right)\log q + \frac{b}{a}\log\frac{b}{a}\right],$$

$$= b\log b - a\log a + b - a.$$

Also solved similarly by P. PENALVER.

MECHANICS.

292. Proposed by C. N. SCHMALL, New York City.

In a bombardment, a battleship directs its fire at a fort standing on a hill whose height is a feet above sea level. The angle of elevation of the fort is found to be ϕ. If the initial velocity of the projectile is v, show that the fort will *not* be struck if $v < \sqrt{ag(1+\csc\phi)}$.

SOLUTION BY PAUL CAPRON, Annapolis, Maryland.

If the projectile is fired at an elevation θ, its trajectory may be represented by $x = vt\cos\theta$, $y = vt\sin\theta - \frac{1}{2}gt^2$, or by

$$y = x\tan\theta - \frac{gx^2}{2v^2}\sec^2\theta.$$

To find the envelope of all such trajectories, we have

$$0 = x\sec^2\theta - \frac{gx^2}{v^2}\sec^2\theta\tan\theta, \qquad \text{or} \qquad \tan\theta = \frac{v^2}{gx};$$

whence

$$y = \frac{v^2}{2g} - \frac{gx^2}{2v^2}$$

is the envelope.

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* To sum

$$q + 2q^2 + 3q^3 + \cdots + (n-1)q^{n-1} = q(1 + 2q + 3q^2 + \cdots + (n-1)q^{n-2}) = q\Sigma$$

$$\Sigma = 1 + 2q + 3q^2 + \cdots + (n-1)q^{n-2}$$

$$-q\Sigma = -q - 2q^2 - \cdots - (n-2)q^{n-2} - (n-1)q^{n-1}$$

$$\therefore (1-q)\Sigma = 1 + q + q^2 + \cdots + q^{n-2} - (n-1)q^{n-1}$$

$$= \frac{q^{n-1}-1}{q-1} - (n-1)q^{n-1}$$

$$\therefore q\Sigma = \frac{q - nq^n + (n-1)q^{n+1}}{(1-q)^2}.$$

The condition that the fort shall not be hit is given by

$$x = a \cot \phi, \qquad a > y;$$

or by

$$a > \frac{v^2}{2g} - \frac{ga^2 \cot^2 \phi}{2v^2} \qquad \text{or} \qquad v^4 - 2agv^2 < a^2g^2 \cot^2 \phi,$$

or

$$(v^2 - ag)^2 < a^2g^2 \csc^2 \phi, \qquad \text{or} \qquad v < \sqrt{ag(1 + \csc \phi)}.$$

Also solved by L. Sivian, F. C. Feemster, A. M. Harding, and Clifford N. Mills.

NUMBER THEORY.

220. Proposed by E. T. BELL, Seattle, Washington.

Let $[m/n]$ denote the greatest integer that is not greater than m/n; and let the two sets,

$$\left[\frac{m}{m-1}\right]; \left[\frac{m+1}{m-2}\right]; \left[\frac{m+2}{m-3}\right]; \cdots \left[\frac{2m-3}{2}\right],$$

$$\left[\frac{m-1}{m-1}\right]; \left[\frac{m}{m-2}\right]; \left[\frac{m+1}{m-3}\right]; \cdots \left[\frac{2m-4}{2}\right],$$

be denoted by (A) and (B) respectively.

Prove that a necessary and sufficient condition that $2m - 1$ be a prime number is that the excess of the number of even integers in (A) over the number of even integers in (B) shall be equal to the excess of the number of odd integers in (A) over the number of odd integers in (B).

Solution by Elijah Swift, University of Vermont.

Unless one of the quantities $m/(m-1)$, $(m+1)/(m-2)$, $\cdots$, $(2m-3)/2$ is an integer the two sets (A) and (B) will be exactly the same. If one of the above quantities is an integer we shall show (1) that it is an even integer; and (2) that $2m - 1$ will then possess a factor.

Let $(m + k)/(m - k - 1) =$ an integer, say α. This is equivalent to the equation $2m - 1 = (\alpha + 1)(m - k - 1)$, which proves the statements above.

Suppose now that $2m - 1$ is prime. Then sets (A) and (B) will be exactly the same. Hence the condition is necessary. If $2m - 1$ is composite, each factor is odd. Call them $\alpha + 1$ and $m - k - 1$, k must be as large as 1. But $2m - 1 = (\alpha + 1)(m - k - 1)$ is the same as $(m + k)/(m - k - 1) = \alpha =$ an integer. Hence, at least one of the integers of (A) is even and the condition can not be fulfilled. Hence the condition is sufficient.

QUESTIONS AND DISCUSSIONS.

Edited by U. G. Mitchell, University of Kansas.

At the time of making up copy for this issue replies had not yet been received for numbers 4, 8, 12, 13, 16, 20 and further replies are desired for numbers 23, 24 and 25. May we have these for the September issue?

REPLIES.

11. In our courses of study is it desirable to give more consideration to vector analysis? What topics should be included in a first treatment of this subject?

Reply by E. L. Rees, University of Kentucky.

The writer is much disappointed that question eleven has not elicited sufficient interest to bring forth a full discussion, such, for example, as that on question three. While he feels that synthetic geometry, as outlined by Professor Bussey, makes a very desirable undergraduate course, he also feels that, among the somewhat large number of other possible courses, vector analysis is fully as practicable and valuable as synthetic geometry.

The following, in the opinion of the writer, are some of the merits of such a course:

(*a*) The student's interest in mathematics is stimulated. He is charmed by the symmetry and beauty of the new notation, and, by frequent translations from the vector language to the scalar, he is constantly reminded of the greater brevity and power of the new symbols.

(*b*) The esthetic element, which is so much in evidence in the higher courses of mathematics and which is too often obscured in the grind of the elementary work, impresses itself easily on the student's mind.

(*c*) The student is made to feel the power and comprehensiveness of a new mathematical instrument in the vector methods, without being overburdened and discouraged by the difficulties of the subject.

(*d*) The very wide range of applications serves at once as a review of a large part of his college work and as a means of introducing certain new subjects. This ability to treat such a large variety of subjects, already familiar to the student, by one uniform method, and to introduce entirely new subjects by this method, gives him a more elevated point of view and tends to broaden his mental vision.

(*e*) The student is relieved of the monotony of his other algebraic work by the freshness of a new symbolism with its many interesting interpretations.

(*f*) It affords a most attractive introduction to the theory of space curves and surfaces.

(*g*) Finally, the difficulties of the subject are not too great for students who elect mathematics as their undergraduate major, if only the algebra and a brief introduction to the calculus of a vector function of a scalar be given.

The writer has taught this subject to juniors at the University of Kentucky for a number of years with very satisfactory results. The course has been given by lectures with collateral reading in Gibbs-Wilson and Coffin.

It is hoped that this imperfect discussion will open the way for a much fuller discussion, either by way of criticism or enlargement on what has been written.

25. In an investigation in physics Mr. Mason E. Hufford, 525 S. Park Ave., Bloomington, Indiana, has need of the values of the Bessel functions $J_0(x)$ and $J_1(x)$ for positive real values of x up to $x = 100$. Have tables been constructed to this extent? What is the most ready means by which the desired values may be computed to any required degree of accuracy?

Reply by S. A. Corey, Hiteman, Iowa.

Values of Bessel functions $J_0(x)$ and $J_1(x)$ may, for small values of x, be obtained by ordinary methods of quadrature from the definite integrals

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos (x \cos \phi)\, d\phi,$$

$$J_1(x) = \frac{x}{\pi}\int_0^{\pi} \sin^2 \phi \cos (x \cos \phi) d\phi,$$

by taking Δx equal to $\pi/9$, $\pi/12$, $\pi/18$, or for greater accuracy still smaller values.

For larger values of x use the asymptotic series

$$J_0(x) = \sqrt{\frac{2}{\pi x}}\left[\cos\left(\frac{\pi}{4} - x\right)\left\{1 - \frac{1^2 \cdot 3^2}{1 \cdot 2(8x)^2} + \frac{1^2 \cdot 2^2 \cdot 3^2}{1 \cdot 2 \cdot 3 \cdot 4(8x)^4} \cdots\right\}\right.$$
$$\left. - \sin\left(\frac{\pi}{4} - x\right)\left\{\frac{1^2}{1 \cdot 8x} - \frac{1^2 \cdot 3^2 \cdot 5^2}{1 \cdot 2 \cdot 3(8x)^3} + \cdots\right\}\right],$$

$$J_1(x) = \sqrt{\frac{2}{\pi x}}\left[\left\{1 - \frac{(4-1)^2(4-3)^2}{1 \cdot 2(8x)^2} + \frac{(4-1)^2(4-3)^2(4-5)^2(4-7)^2}{1 \cdot 2 \cdot 3 \cdot 4(8x)^4} - \cdots\right\}\right.$$
$$\left.\times \cos\left(\frac{3\pi}{4} + x\right) - \left\{\frac{4}{1 \cdot 8x} + \frac{(4-1)^2(4-3)^2(4-5)^2}{1 \cdot 2 \cdot 3(8x)^3} + \cdots\right\}\sin\left(\frac{3\pi}{4} - x\right)\right].$$

These asymptotic series are of such a nature that if we stop the calculations with any term the resulting error is less than the next term. As x increases these series become rapidly convergent, so that any degree of accuracy required in any physical investigation is readily obtainable.

NOTES AND NEWS.

Edited by W. D. Cairns.

"The mathematical analysis of electrical and optical wave-motion" is the title of a book by Dr. Harry Bateman just published by G. P. Putnam's Sons.

Professor H. S. White of Vassar College gave an address on March 5 under the auspices of the Syracuse University chapter of Sigma, his subject being "Mathematics in nineteenth century science."

In *Science* for April 2 President R. S. WOODWARD develops further the differences between his treatment of the deviations of freely falling bodies and the methods of Professor F. R. MOULTON and Professor W. H. ROEVER. (See this MONTHLY of December, 1913, and September, 1914.)

Professor C. W. McCORD, who retired from active service at the Stevens Institute of Technology in 1906, where he had been professor of mechanical drawing since 1871, died in Hoboken, N. J., on April 13, 1915. He was the author of a book on descriptive geometry and was justly honored as the draughtsman who with Captain John Ericsson in 1859 made the plans for the *Monitor*.

Professor R. D. CARMICHAEL, of Indiana University, and Professor ARNOLD DRESDEN, of the University of Wisconsin, will be members of the staff in mathematics at the University of Chicago during the summer quarter, 1915. Professors L. E. DICKSON and H. E. SLAUGHT will be absent on vacation.

DR. DANIEL BUCHANAN, of Queen's University, Kingston, Ontario, has been promoted to an associate professorship in mathematics and has been appointed director of the college observatory.

Professor C. J. KEYSER, of Columbia University, and H. W. TYLER, of the Massachusetts Institute of Technology, will lecture this summer at the University of California, the former on "History and significance of central mathematical ideas and doctrines," the latter on "History of mathematical science."

PROFESSOR H. S. WHITE, of Vassar College, was elected a member of the National Academy of Sciences at the annual meeting held last April in Washington, D. C. The other members representing pure mathematics in this Academy are Professors BÔCHER, BOLZA, DICKSON, MOORE, OSGOOD, STOREY, and VAN VLECK.

MR. GEORGE RUTLEDGE, of the University of Illinois, has accepted an instructorship in mathematics in the Massachusetts Institute of Technology.

At the University of Kansas, Dr. U. G. MITCHELL has been promoted to an associate professorship, and C. H. ASHTON to a professorship in the department of mathematics. Dr. E. B. STOUFFER, who went there last year from the University of Illinois, is an assistant professor in the department.

An article on "The place of the textbook in mathematical teaching" by Mr. EDMUND LIGHTLEY appeared in the *School World* for January.

Professor de Morgan's "Budget of Paradoxes," long since out of print, has been edited by Professor DAVID EUGENE SMITH with extended biographical, historical and explanatory notes. It will appear soon from the press of the Open Court Publishing Company.

The cover for the May issue was dated " April " by an oversight of the printer. The printed slip herewith may be used to correct the error. The next issue of the MONTHLY, will be for September, 1915.

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HISTORY OF ZENO'S ARGUMENTS ON MOTION:

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

By FLORIAN CAJORI, Colorado College.

VIII.

D. VIEWED IN THE LIGHT OF AN IDEALISTIC CONTINUUM.

The full explanation of Zeno's paradoxes requires two ideas which are very familiar to the modern mathematician, namely, the acceptance of the existence of actually infinite aggregates and the idea of a connected and perfect continuum. The first concept, that of actual infinity, as opposed to potential infinity, had been under contemplation ever since the time of Aristotle. Men like St. Augustine, Galileo, Pascal, Volder, Schultze seemed to have had a more or less non-contradictory conception of it. Others only denied it. Among the latter were both philosophers and mathematicians; the list includes men like Thomas Aquinas, Gerdil, Descartes, Spinoza, Leibniz, Lock, Lotze, Renouvier, Moigno, Cauchy, Gauss, and of course many others.[1] Wonderful insight into this matter was possessed by Galileo. He showed for instance that there were as many integers that were perfect squares as there were integers altogether. Strange to say, Galileo's argument has been misinterpreted by some recent writers. Instead of accepting the conclusion as legitimate for that sort of infinity, as it was accepted by Galileo, these writers declared the conclusion absurd, hence the hypothesis of actual infinity a myth. Among men taking this view was the French philosopher F. Pillon.[2] In 1831 (July 12) the great K. F. Gauss of Göttingen wrote a letter to Schumacher in which he declares himself as opposed to the actual infinity in mathematics.

[1] For details see Georg Cantor, "Ueber die verschiedenen Standpunkte in Bezug auf das actuele Unendliche" in *Zeitschr. f. Philos. u. philos. Kritik*, Bd. 88, p. 224.

[2] *L'année philosophique*, I, 1890, p. 84, quoting among others Cauchy's *Sept leçons de physique générale*, 1868, as follows: "Cette proposition fondamentale, démontrée par Galilée (qu'on ne saurait admettre une suite ou série actuellement composée d'un nombre infini de termes), s'applique aussi bien à une série de termes ou d'objects, qui ont existé." Cauchy was strongly influenced on this matter by the writings of Gerdil.

In the nineteenth century voices in favor of the actual infinity began to speak with greater emphasis. In 1823 John Bolyai, of non-euclidean geometry fame, wrote down that quality of an infinite aggregate: "An infinite aggregate is one equivalent to a part of itself."[1] Another pioneer in this field was the Bohemian mathematician, Bernard Bolzano, whose writings have only in recent years begun to receive proper appreciation.[2] After the appearance of Cantor's writings his ideas received wide recognition among mathematicians, notwithstanding certain perplexing paradoxes to which some of the more advanced developments of the subject gave rise. Among philosophers the Cantorian ideas found slower recognition. The question at issue is usually not so much one of logic, as it is of the postulates which the reasoner is willing to accept as reasonable and useful. An investigator who vetoes any assumption which does not appeal to his intuition or to his power of imagination can hardly find comfort in Cantor's theory of aggregates and the Cantor continuum. To him Zeno's paradoxes must necessarily remain paradoxes forever.

The second notion needed for the full elucidation of our subject is the "connected" and "perfect" continuum, which we owe to Georg Cantor and Dedekind. To their names should be added that of Karl Weierstrass who banished from analysis the mystical notion of the infinitesimal as a constant smaller than any assignable number, defying the Archimedian postulate. We are not aware that any of these three men wrote directly on the paradoxes of Zeno. But they laid the foundation on which a rational theory of them rests. Richard Dedekind brought out two wellknown publications: *Stetigkeit und irrationale Zahlen*, Braunschweig, 1872, and *Was sind und was sollen die Zahlen*, Braunschweig, 1888. Georg Cantor's first important publication on the theory of aggregates is his *Grundlagen einer allgemeinen Mannichfaltigkeitslehre*, Leipzig, 1883. The fundamental ideas advanced by Dedekind and Georg Cantor are so easily accessible and so generally known, that no account of them is needed here. At first British mathematicians took little interest in Cantor's developments. Only in recent years have they been taken up in Great Britain. An unusually interesting outline of them is given by Ernest William Hobson in his presidential address "On the Infinite and the Infinitesimal in Mathematical Analysis," before the London Mathematical Society, in 1902.[3] We quote the following:

"When it is conceived that these mere potentialities pass into actualities, that *fixed* numbers or magnitudes exist which are infinite or infinitesimal, that the mere indefinitely great becomes an actual infinite, or the merely indefinitely small becomes an actual infinitesimal, the region of serious controversy has been reached. . . .

"Here we have the origin of the method of limits, in its geometrical and its arithmetical forms, and here we come across the central difficulty of the mode in which a limit was regarded as being actually attained. A limit which appeared only as the unattainable end of a process of indefinite regression, and to which unending approach was made, had, by some process inaccessible to the sensuous imagination, to be regarded as actually reached; the chasm which separated the limit from the approaching magnitudes had in some mysterious way to be leapt over. . . .

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[1] See Halsted's *Bolyai's Science Absolute of Space*, 4th Ed., 1896, § 24, p. 20.

[2] See H. Bergmann, *Das philosophische Werk Bernard Bolzanos*, Halle, 1909; also F. Přihonský, *Dr. Bernard Bolzano's Paradoxien des Unendlichen*, Berlin, 1889.

[3] *Proceedings London Mathematical Society*, Vol. 35, London, 1903, p. 117.

"The notion of number, integral or fractional, has been placed upon a basis entirely independent of measurable magnitude, and pure analysis is regarded as a scheme which deals with number only, and has, *per se*, no concern with measurable quantity. Analysis thus placed upon an arithmetical basis is *characterized by the rejection of all appeals to our special intuitions of space, time and motion, in support of the possibility of its operations.* . . ."

"By this conception of the domain of number the root difficulty of the older analysis as to the existence of a limit is turned, each number of the continuum being really defined in such a way that it itself exhibits the limit of certain classes of convergent sequences. . . . It should be observed that the criterion for the convergence of an aggregate is of such a character that no use is made in it of infinitesimals, definite finite numbers alone being used in the test. . . .

"This [old] intuitive notion of the continuum appears to have as its content the notion of unlimited divisibility, the facts that, for instance, in the linear continuum we can within any interval PQ find a smaller one, $P'Q'$, that this process may be continued as far as the limits of our perception allow, and that we are unable to conceive that even beyond the limits of our perception the process of divisibility in thought can come to an end. However, the modern discussions as to the nature of the arithmetic continuum have made it clear that this property of unlimited divisibility, or connexity, is only one of the distinguishing characteristics of the continuum, and is insufficient to mark it off from other domains which have the like property. The aggregate of rational numbers, or of points on a straight line corresponding to such numbers, possess this property of connexity in common with the continuum, and yet it is not continuous." . . .

"The other property of the aggregate which is characteristic of the continuum, is that of being, in the technical language of the theory of aggregates (Mengenlehre) perfect: the meaning of this is that all the limits of the converging sequences of numbers or parts belonging to the aggregate themselves belong to the aggregate; and, conversely, that every number or point of the aggregate can be exhibited as the limit of such a sequence. . . .

". . . the latter property of the continuum, which was not brought to light by those who took the intuitive continuum as a sufficient basis, is in some respects the more absolutely essential property for the domain of a function which is to be submitted to the operation of the calculus."

"In order to exhibit the way in which transfinite ordinal numbers are required when we deal with non-finite aggregates, I propose to refer to a well-known paradox of Achilles and the tortoise. . . . Let us indicate the successive positions of Achilles referred to, by the ordinal numbers 1, 2, 3, . . . suffixed to the letter A, so that $A_1\ A_2\ A_3$ . . . represent the positions of Achilles. . . . These points $A_1\ A_2$

$$\begin{array}{llllllll} B_2 & B_3 & B_4 & & & & & \\ \hline A_1 & A_2 & A_3 & A_4 & A_\omega & A_{\omega+1} & A_{\omega+2} & A_{\omega 2} \end{array}$$

A_3 . . . have a limiting point, which represents the place where Achilles actually catches the tortoise. The limiting point is not contained in the sets of points $A_1\ A_2\ A_3$ . . .; if we wish to represent it, we must introduce a new symbol ω, and denote the point by this number. It does not occur in the series 1, 2, 3, . . . but is preceded by all of these numbers, and yet there is no number immediately preceding it; it is the first of a new series of numbers."

Hobson proceeds to show how a finite number can maintain itself against a transfinite ordinal number, by showing that $\omega = 1 + \omega$, but $\omega + 1 > \omega$; the commutative law in addition is seen to fail. He brings out the necessity for the introduction of transfinite numbers for the representation of the limit which is not itself contained within the region of the convergent process. Hobson's exposition represents an explanation which recent developments of mathematics offer of the "Dichotomy" and the "Achilles." No doubt, some readers might have desired a fuller exposition of details. It will be noticed that the time-element was not considered by Hobson at all. The Dedekind and Georg Cantor theories of the continuum do not involve the element of time. But how is it possible to ignore time in questions involving motion? In the first place it is pointed out by Cantor[1] that the continuum is a much more primitive and general

[1] *Grundlagen einer allgemeinen Mannichfaltigkeitslehre*, von Georg Cantor, Leipzig, 1883, p. 29.

concept than the concept of time, that the theory of the continuum is needed for a clear exposition of time or of any independent variable, that time cannot be considered as the measure of motion; on the contrary, time is measured by motion—the motions of heavenly bodies, the motions of the hands of a watch or clock, the displacement of sand in the hour glass. In the second place the consideration of time is not needed at the critical point where the ability of Achilles to overtake the tortoise is under consideration. Suppose the tortoise has an initial start of 10 ft. and that it travels 1 ft. per second, while Achilles travels 10 ft. per second. Forming the series A, whose terms represent each the distance Achilles travels to come up to the place where the tortoise was at the beginning of the time interval under consideration, and letting the series T represent these time-intervals, we have

$$A \qquad 10 + 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots,$$

$$T \qquad 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \cdots.$$

Both geometric infinite series are convergent; the sum of the terms of each series approaches a finite number as a limit. Now comes the ever-present, delicate question whether the sum actually *reaches* its limit. It is to be observed that this question arises in each series, that one series does not help the other. If the sum of A reaches its limit, so does the sum of T, but the possibility of the sum of A reaching its limit is a consideration independent of T. In this sense the consideration of time does not enter the critical part in the explanation of the "Achilles." Whether the sum of A reaches its limit or not is a matter of pure assumption on our part. If the limiting value $11\frac{1}{9}$ ft. is assumed to be included in the aggregate of numbers which the distance-variable may take, then of course the variable reaches its limit; if $11\frac{1}{9}$ ft. is not assumed as a value which the variable may take, then of course the limit is not reached. It is here that we must receive a suggestion from our sensuous observations; we know from our knowledge of motion as supplied to us by our senses that Achilles travelling with a uniform finite velocity in the same direction will within a finite time reach the distance $11\frac{1}{9}$ ft. from his starting point. This information, supplied to us by our senses, enables us to choose, of the two possible alternative assumptions offered by theory (as mentioned above), the one which makes the variable sum A conform with the known sensuous phenomena. On this assumption the region of the convergent process has a limit which is contained in the aggregate of values the variable can take, and, as explained by Hobson, the limit is *reached by the variable*. Viewed from the standpoint of the theory of infinite aggregates and of the Georg Cantor conception of the continuum, the "Achilles" is almost a self-evident proposition. Sensuous knowledge suggests that we make the aggregate of values of the variable distance travelled by Achilles a *perfect* aggregate; then theory tells us that in a perfect aggregate every converging process has a limit which is reached.

Interesting remarks on the Georg Cantor continuum are found also in Hobson's *Theory of Functions of a Real Variable*, Cambridge, 1907, p. 51:

"The term 'arithmetic continuum' is used to denote the aggregate of real numbers, because it is held that the system of numbers of this aggregate is adequate for the complete analytical representation of what is known as continuous magnitude. The theory of the arithmetic continuum has been criticised on the ground that it is an attempt to find the continuous within the domain of number, whereas number is essentially discrete. Such an objection presupposes the existence of some independent conception of the continuum, with which that of the aggregate of real numbers can be compared. At the time when the theory of the arithmetic continuum was developed the only conception of the continuum which was extant was that of the continuum as given by intuition: but this, as we shall show, is too vague a conception to be fitted for an object of exact mathematical thought, until its character as a pure intuitional datum has been modified by exact definitions and axioms."

It will be seen as we proceed that the objection to the Georg Cantor continuum, to which Hobson refers, is frequently made. It is the general objection that a line which is continuous cannot possibly be constructed out of mathematical points, external to each other. Perhaps this general objection which naturally suggests itself at the very start has discouraged non-mathematicians from going to the trouble of studying the Cantor continuum with the care necessary for its comprehension. Philosophers who have subjected themselves to such study have been amply repaid for their labor. They have found it to be a device of the understanding "whereby we give conceptual unity and an invisible connectedness to certain types of phenomenal facts which come to us in a discrete form and in a confused variety."[1]

The other important shift in the point of view, made by the creators of the modern linear continuum, was the rejection of all infinitesimals, that is of quantities which do not obey the Archimedian postulate. This postulate says that if a and b are two numbers (not zero), such that $a < b$, then it is always possible to find a finite integer n so that $na > b$. The infinitesimal which had been the subject of many controversies and was regarded by many as containing an element of mysticism, was banished by Weierstrass and Cantor from their mathematical concepts. In former years the infinitesimal was considered as necessary in the explanation of the linear continuum. Johann Heinrich Lambert wrote to Holland in a letter of April 7, 1766, on the "angle of contact" as follows:[2]

"Do you believe, my dear Sir, that one can dispense with the concept of the infinitely small in the concept of continuity? . . . Continuity demands that this variation be less than every assignable quantity. It is thus impossible to estimate this variation by a finite quantity, and equal to 0 it can not be either. There seems therefore nothing left than to say that the change in direction is infinitely small."

The impossibilities of one generation often become the possibilities of a succeeding generation. Weierstrass's banishment of the infinitely small has found wide following; the old-time infinitesimal is no longer needed in explaining the continuum. The rejection of the infinitely small is looked upon by such mathe-

[1] H. Poincaré, *The Foundations of Science*, transl. by G. B. Halsted, New York, 1913, introduction by Josiah Royce, p. 16.

[2] *J. H. Lamberts deutscher gelehrter Briefwechsel*, Vol. I, Berlin, 1781, p. 141.

matical philosophers and logicians as Bertrand Russell[1] and A. N. Whitehead[2] as steps toward greater mathematical rigor. It must be emphasized, however, that the school of Weierstrass has not found universal recognition; there are modern champions of the infinitely small, chief among whom is the Italian mathematician Giuseppe Veronese. They insist that the Cantor continuum is not the only possible non-contradictory continuum and proceed to construct a higher and more involved, non-archimedean, continuum in which infinitely small distances are given. This is not the place for attempting a minute statement of the controversy between the two schools; the controversy, by the way, has no national aspect. There have been followers of Veronese in Germany (for instance, Stolz, Max Simon), and followers of Weierstrass and G. Cantor in Italy (for instance, Peano). So far as we have noticed, the Zeno arguments have not been studied and given explicit treatment on the basis of the Veronese continuum.[3] In America C. S. Peirce has adhered to the idea of infinitesimals in the declaration: "The illumination of the subject by a strict notation for the logic of relatives had shown me clearly and evidently that the idea of an infinitesimal involves no contradiction."[4] Apparently, before he had acquired familiarity with the writings of Dedekind and Georg Cantor, C. S. Peirce had firmly recognized that for infinite collections the axiom, that the whole is greater than its part, does not hold.

[To be continued]

ON NAPIER'S FUNDAMENTAL THEOREM RELATING TO RIGHT SPHERICAL TRIANGLES.

By ROBERT MORITZ, University of Washington.

In view of the recent celebration of the tercentenary of the publication of Napier's greatest work, the "Mirifici logarithmorum canonis descriptio," it is highly fitting that his rule for the circular parts should be rescued from the rubbish heap of mnemotechnics and be assigned its proper place as the most

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[1] See, for instance, his article in the *International Monthly*, Vol. 4, 1901, p. 84 and seq.

[2] A. N. Whitehead, *Introduction to Mathematics*, New York and London, 1911, pp. 156, 226–229.

[3] References to this controversy are as follows: G. Veronese, *Grundzüge der Geometrie von mehreren Dimensionen*, übersetzt v. A. Schepp, Leipzig, 1894, Anhang, p. 631–701; Max Simon, "Historische Bemerkungen über das Continuum, den Punkt und die Gerade Linie," *Atti del IV. Congresso Internazionale dei matematici*, Roma, 1908, pp. 385–390; G. Cantor's letter to Vivanti, *Rivista di mat.* V, 104–108; G. Cantor's letter to Peano, *Rivista di mat.* V, 108–109; G. Cantor, "Zur Begründung der Transfiniten Mengenlehre I," *Mathematische Annalen*, Vol. 46, 1895, page 500; Frederico Enriques, *Probleme der Wissenschaft*, 2. Teil, übersetzt von K. Grelling, Leipzig und Berlin, 1910, pp. 324–329. An able discussion of infinity, infinitesimals and the continuum is given by Josiah Royce, a philosopher familiar with mathematical thought, in his *The World and the Individual*, New York, 1900, pp. 505–560. See also G. Cantor, "Mitteilungen zur Lehre vom Transfiniten" in *Zeitsch. für Philosophie u. Philosophische Kritik*, Vol. 91, Halle, 1887, p. 113; O. Stolz in *Mathematische Annalen*, Bd. XVIII, p. 699, also in *Berichte des naturw.-medizin. Vereins in Innsbruck*, Jahrgänge 1881–82 und 1884, also in *Vorlesungen über allgem. Arithm.*, Leipzig, 1. Theil, 1885, p. 205.

[4] C. S. Peirce, "The Law of Mind" in *The Monist*, Vol. 2, 1892, p. 537.

beautiful theorem in the whole field of elementary trigonometry. It is one of the strange vicissitudes of fortune that the elegant proof which was clearly indicated by Napier himself in the fourth chapter of the second book of the "descriptio" and rediscovered by Lambert[1] and Ellis[2] should nevertheless have remained generally unknown to writers on trigonometry in the nineteenth century and that even to this day the impression generally prevails that Napier's rules are nothing more than mnemonic devices whose utility as an instrument may well be questioned. Excepting two recent texts[3] one seeks in vain for any intimation that Napier's rules for the circular parts have any other than an inductive basis.

Thus as omniscient a writer as DeMorgan in his *Spherical Trigonometry* speaks of Napier's rules as "mnemonical formulas" and expresses his conviction that they "only create confusion instead of assisting the memory." Chauvenet (*Plane and Spherical Trigonometry,* 1891) after developing the ten formulas for the solution of right spherical triangles, says: "By putting these ten rules under a different form, Napier contrived to express them all in two rules, which, though artificial, are very generally employed as aids to the memory." In like tenor Newcomb (*Trigonometry,* 1893): "The six preceding formulæ, which may be found difficult to remember, have been included by Napier in two precepts of remarkable simplicity, and easily remembered" and the same view is reiterated in the recent work of Bôcher and Gaylord (*Trigonometry,* 1914) in the words "Formulas 1–10 may be collected into a very compact and convenient form by means of a rule formulated by John Napier. The student should prove that these rules are correct by applying them in succession to all five parts of the figure."

Nor is the impression that Napier's rules have no other than an inductive basis limited to writers of textbooks on trigonometry. Cajori in his *History of Mathematics* states that "Napier's Rule of circular parts is perhaps the happiest example of artificial memory that is known," thus putting this remarkable achievement in deduction on a par and in direct competition with the famous mnemonic hexameter of the logicians,

> "Barbara, Celarent, Darii, Ferioque prioris.
> Cesare, Camestres, Festino, Baroko secundae.
> Tertia Darapti, Disamis, Datisi, Felapton,
> Bokardo, Ferison habet. Quarta insuper addit
> Bramantip, Camenes, Dimaris, Fesapo, Fresison,"

of which Hamilton said "there are few human inventions which display a higher ingenuity."

Even as high and recent an authority as E. W. Hobson in his article on trigonometry in the eleventh edition of the Encyclopedia Britannica dismisses the whole matter with the words "Napier gave mnemonical rules for remembering" the right spherical triangle relations.

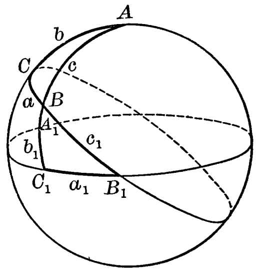
[1] *Beiträge zur Mathematik,* I (1765), p. 375 et seq.

[2] *The Mathematical and other Writings of Robert Leslie Ellis* (1863), p. 328 et seq.

[3] Leathem-Todhunter: *Spherical Trigonometry,* Macmillan (1907). Moritz: *Spherical Trigonometry,* Wiley and Sons (1914).

In view of this generally prevailing misconception, it may be timely to present in this place an elementary proof of Napier's theorem which will in a sense supplement Professor E. O. Lovett's prior note[1] on the same subject.

Let $\Delta = ABC$ be a right spherical triangle whose parts taken in counter-clockwise order beginning with a side adjacent to the right angle are a, B, c, A, b. With A as a pole construct the arc of a great circle intersecting the arcs CB produced and AB produced in B_1 and C_1 respectively. A second right triangle $\Delta_1 = A_1B_1C_1$ is thus formed whose parts a_1, B_1, c_1, A_1, b_1 are



$$a_1 = \bar{A},\ B_1 = \bar{b},\ c_1 = \bar{a},\ A_1 = B,\ b_1 = \bar{c},$$

where

$$\bar{A} = 90° - A, \qquad \bar{b} = 90° - b, \qquad \bar{a} = 90° - a, \text{ etc.}$$

Thus every triangle Δ whose circular parts are $a\ \bar{B}\ \bar{c}\ \bar{A}\ b$, leads to a second triangle Δ_1 whose circular parts are $\bar{A}\ b\ a\ \bar{B}\ \bar{c}$, and this likewise to a third Δ_2 whose circular parts are $\bar{B}\ \bar{c}\ \bar{A}\ b\ a$, and this again to a fourth Δ_3 whose circular parts are $b\ a\ \bar{B}\ \bar{c}\ \bar{A}$, and this in turn to a fifth Δ_4 whose circular parts are $\bar{c}\ \bar{A}\ b\ a\ \bar{B}$. The fifth triangle, Δ_4, leads to the first, Δ, thus completing the cycle.

The circular parts of these five triangles are thus shown to be the same for all. Furthermore, if we call the hypotenuse of a right triangle its middle part, this name may be applied to each of the circular parts, for each of the circular parts $a, \bar{B}, \bar{c}, \bar{A}, b$, is the hypotenuse of some one of the five triangles Δ, Δ_1, Δ_2, Δ_3, Δ_4. It follows that the relation between any middle part and its two adjacent (contiguous) parts applies to every middle part and its two adjacent parts, and again, that the relation between any middle part and its two opposite (non-contiguous) parts applies to every middle part and its two opposite parts.

Now from the first triangle ABC we find from the law of cosines (or otherwise) that $\cos c = \cos a \cos b$, that is $\sin \bar{c} = \cos a \cos b$, and on eliminating from this equation a and b by the law of sines (or otherwise) we obtain $\cos c = \cot A \cot B$, that is $\sin \bar{c} = \tan \bar{A} \tan \bar{B}$. This proves *Napier's Theorem: The sine of any circular part is equal to the product of the cosines of the opposite parts and to the product of tangents of the adjacent parts.*

[1] *Bulletin of the American Mathematical Society,* 1898, p. 552.

AN INTERPOLATION FORMULA FOR POISSON'S EXPONENTIAL BINOMIAL LIMIT.

By E. C. MOLINA, New York City.

Number LII of the very valuable "Tables for Statisticians and Biometricians" recently edited by Karl Pearson gives the values of Poisson's exponential binomial limit,

$$P(c, a) = \sum_{s=c}^{\infty} \frac{\epsilon^{-a} a^s}{\underline{|s}}$$

for integral values of a from 1 to 30 inclusive.

Suppose we want the value of

$$P(c, a + \delta), \qquad (\delta = 0.1, 0.2, \cdots 0.9).$$

By successive partial integration it is easy to see that

$$P(c, a) = \frac{1}{\underline{|c-1}} \int_0^a x^{c-1} \epsilon^{-x} dx. \tag{1}$$

Therefore,

$$P(c, a + \delta) = P(c, \delta) + \frac{1}{\underline{|c-1}} \int_\delta^{a+\delta} x^{c-1} \epsilon^{-x} dx. \tag{2}$$

Now

$$\begin{aligned} \frac{1}{\underline{|c-1}} \int_\delta^{a+\delta} x^{c-1} \epsilon^{-x} dx &= \frac{\epsilon^{-\delta} \delta^{c-1}}{\underline{|c-1}} \int_0^a \left(1 + \frac{y}{\delta}\right)^{c-1} \epsilon^{-y} dy \\ &= \sum_{u=1}^{c} \frac{\epsilon^{-\delta} \delta^{c-u}}{\underline{|c-u}\,\underline{|u-1}} \int_0^a y^{u-1} \epsilon^{-y} dy \qquad (3) \\ &= \sum_{u=1}^{c} \left[\frac{\epsilon^{-\delta} \delta^{c-u}}{\underline{|c-u}} P(u, a) \right] = \sum_{t=0}^{c-1} \left[\frac{\epsilon^{-\delta} \delta^t}{\underline{|t}} P(c-t, a) \right]. \end{aligned}$$

Thus, finally

$$P(c, a + \delta) = P(c, \delta) + \sum_{t=0}^{c-1} \left[\frac{\epsilon^{-\delta} \delta^t}{\underline{|t}} P(c-t, a) \right]. \tag{4}$$

As stated above, the values of $P(c - t, a)$ for $a \not> 30$ are given in Pearson's Table LII. Also, in the first section of Pearson's Table LI are given the values of $\epsilon^{-\delta} \delta^t / \underline{|t}$.

For the practical application of formula (4) we need, however, the values of $P(c, \delta)$. These are given in the short tables which follow.

Table Giving Values of $P(c, \delta)$.

c	δ									
	0.1	0 2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	......	......	......	......	......	......	......	......	......	.....
1	.095163	.181269	.259182	.329680	.393469	.451188	.503415	.550671	.593431	.63212
2	.004679	.017523	.036936	.061552	.090204	.121901	.155805	.191208	.227518	.26424
3	.000155	.001148	.003600	.007926	.014388	.023115	.034142	.047423	.062857	.08030
4	.000004	.000057	.000266	.000776	.001752	.003358	.005753	.009080	.013459	.01899
5	.000000	.000002	.000016	.000061	.000172	.000395	.000786	.001411	.002344	.00366
6	......	.000000	.000001	.000004	.000014	.000039	.000090	.000184	.000343	.00059
7	......	......	.000000	.000000	.000001	.000003	.000009	.000021	.000043	.00008
8	......	......	......	......	.000000	.000000	.000001	.000002	.000005	.00001
9	......	......	......	......	......	......	.000000	.000000	.000000	.00000

Example.—Find value of P (20, 14.8).
Here $c = 20$, $a = 14$, $\delta = .8$.

t	$\epsilon^{-\delta}\delta^t/\lfloor t$, Table LI, page 113.	$P(c-t, a)$, Table LII, page 123.	$\frac{\epsilon^{-\delta}\delta^t}{\lfloor t} P(c-t, a)$.
0	.449329	.07650	.034374
1	.359463	.11736	.042186
2	.143785	.17280	.024846
3	.038343	.24408	.009359
4	.007669	.33064	.002536
5	.001227	.42956	.000527
6	.000164	.53555	.000088
7	.000019	.64154	.000012
8	.000002	.73996	.000001
9	.000000		.113929

$$P(c, \delta) = P(20,\ 0.8) = .000000$$
$$P(c, a + \delta) = P(20,\ 14.8) = .11393$$

January 19, 1915.

A SIMPLE SOLUTION OF THE DIOPHANTINE EQUATION $U^3 = V^3 + X^3 + Y^3$.

By J. W. NICHOLSON, Louisiana State University.

There are many sets of four integers such that the cube of the first is equal to the sum of the cubes of the other three. For examples,

$$6^3 = 5^3 + 4^3 + 3^3, \tag{1}$$

$$9^3 = 8^3 + 6^3 + 1^3, \tag{2}$$

$$19^3 = 18^3 + 10^3 + 3^3. \tag{3}$$

The purpose of this paper is to deduce a formula by which an unlimited number of such sets may be obtained. It is an interesting question and has frequently engaged the attention of mathematicians. The problem has been treated by Diophantus, Fermat, Euler, and others. Notwithstanding their work, it is believed that the following solution may be of interest. Furthermore, by sub-

stantially the same method of deduction we may solve many other problems in indeterminate analysis, notably the following,

$$r^3 = s^3 + t^3 + u^3 + v^3,$$

$$r^3 = s^3 + t^3 + u^3 + v^3 + x^3,$$

and so on, for any greater number of terms whatever.

Problem. Given one set of four integers m, n, p, r such that

$$m^3 = n^3 + p^3 + r^3, \tag{4}$$

to find a formula for an indefinite number of such sets. Evidently,

$$(my)^3 = (ny)^3 + (py)^3 + (ry)^3 \tag{5}$$

where y may have any value whatever.

Assume

$$(my - bx)^3 = (ny - bx)^3 + (py - ax)^3 + (ry + ax)^3, \tag{6}$$

in which a and b are arbitrary constants.

We wish now to find such values of x and y as will make (6) an identity. Expanding in (6) and reducing, we find

$$(mb^2 - nb^2 - pa^2 - ra^2)x = (m^2b - n^2b - p^2a + r^2a)y.$$

We may therefore take

$$y = mb^2 - nb^2 - pa^2 - ra^2; \quad x = m^2b - n^2b - p^2a + r^2a.$$

Substitute these values for x and y in (6) and we obtain the required formula:

$$\begin{aligned}[(mp + mr)a^2 - (p^2 - r^2)ab + (mn - n^2)b^2]^3 \\ = [(np + nr)a^2 - (p^2 - r^2)ab + (m^2 - mn)b^2]^3 \\ + [(pr + r^2)a^2 + (m^2 - n^2)ab - (mp - np)b^2]^3 \\ + [(p^2 + pr)a^2 - (m^2 - n^2)ab - (mr - nr)b^2]^3.\end{aligned} \tag{A}$$

For example we may make $m = 6$, $n = 5$, $p = 4$, $r = 3$, and obtain

$$\begin{aligned}(42a^2 - 7ab + 5b^2)^3 = (35a^2 - 7ab + 6b^2)^3 \\ + (28a^2 - 11ab - 3b^2)^3 + (21a^2 + 11ab - 4b^2)^3,\end{aligned} \tag{B}$$

which holds for all values of a and b.

Thus, for $a = 10$ and $b = 1$, $4135^3 = 3436^3 + 2687^3 + 2206^3$.

Again, for $a = 1$ and $b = 4$, $103^3 = 94^3 + 64^3 - 1^3$.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY, University of Minnesota.

Vocational Mathematics. By WILLIAM H. DOOLEY. D. C. Heath and Co., Boston, 1915. 358 pages. $1.00.

The author of this book is the principal of the Technical High School of Fall River, Mass. He states in the preface that, in his ten years of experience in organizing and conducting intermediate and secondary technical schools, he has noticed the inability of the regular teachers in mathematics to give the pupils the training in commercial and rule of thumb methods of solving mathematical problems that are so necessary in everyday life. He says that the pupils graduate from the course in mathematics without being able to "commercialize" or apply their mathematical knowledge in such a way as to meet the needs of trade and industry. It was to overcome this difficulty that he wrote this book. It is in the following ten parts, whose names sufficiently indicate the scope of the work: Review of arithmetic; carpentering and building; sheet metal work; bolts, screws and rivets; shafts, pulleys and gears; plumbing and hydraulics; steam engineering; electrical work; mathematics for machinists; textile calculations. There is an appendix of about 40 pages devoted to the metric system, graphs, formulas, logarithms, trigonometry, and tables of various kinds.

W. H. BUSSEY.

A Review of Algebra. By ROMEYN HENRY RIVENBURG. American Book Co., New York, 1914. 80 pages.

This is a book of problems prepared by the head of the department of mathematics of the Peddie Institute, Hightstown, N. J. It begins with a seven-page outline of elementary and intermediate algebra which contains important definitions, special rules for multiplication and division, cases in factoring, etc. It is designed for senior high-school students who need a thorough reivew of algebra in order to prepare for college entrance examinations and for effective work in the freshman year in college. The whole scheme of the book is ordinarily to have a page of problems represent a day's work. It includes quadratics, simultaneous quadratics, the progressions and the binomial theorem. There are twenty-three pages of actual college entrance examinations at the end of the book. The author states in the preface that the student is expected to use his regular text book in algebra for reference, as he would use a dictionary,—to recall a definition, a rule, or a process that he has forgotten.

W. H. BUSSEY.

An Introduction to Laboratory Physics. By LUCIUS TUTTLE. Jefferson Laboratory of Physics, Philadelphia, 1915. 150 pages.

This book is essentially a revision of the mimeographed direction sheets that have been used in the first part of the laboratory course given by the author at

Jefferson Medical College. It does not cover the ground of the usual laboratory manual of physics, but is intended to precede the use of any one of them in a course of physical measurements. It contains chapters on weights and measures, angles and circular functions, accuracy and significant figures, logarithms, small magnitudes, the slide rule, graphic representation, graphic analysis, the principle of coincidence, measurements and errors, statistical methods, deviation and dispersion, the weighting of observations, criteria of rejection, and least squares and various errors.

W. H. Bussey.

University of Minnesota.

Resistance of Materials. For beginners in Engineering. By S. E. Slocum, Professor of Applied Mathematics in the University of Cincinnati. Ginn & Company, Boston, 1914. $2.00.

The chief feature which distinguishes this volume from other American text-books on the same subject, as stated by the author in his preface, is that the principle of moments is used consistently throughout in place of the usual calculus processes.

The subject matter is presented in a clear and concise manner and is written so the book may be used by students at the same time they are taking courses in calculus or even before taking such courses. This feature makes the book available for trade or architectural schools where no calculus is taught.

The text has been divided into the following fourteen sections: stress and deformation; first and second moments; bending moment and shear diagrams; strength of beams; deflection of cantilever and simple beams; continuous beams; restrained or built-in beams; columns and struts; torsion; spheres and cylinders under uniform pressure; flat plates; riveted joints and connections; reinforced concrete; simple structures.

The author has used the principle of integration freely throughout the text dealing with the small parts, "elements," represented by Δ and using the symbol Σ instead of $\int$. Such a ratio as $\Delta M/\Delta x$ has been defined as the "rate of change," instead of the usual dM/dx of the calculus; also in finding the area of the moment diagram the expression $\Sigma M \cdot \Delta x$ replaces the $\int Mdx$ of the calculus.

Instead of deducing the equation of the elastic curve of a loaded beam and from it the slope and deflection at any section, the author has used for the deflection, $d = 1/EI$ (static moment of the moment diagram), and for the slope, $\tan \varphi = A/EI$, where A represents the area of the moment diagram between any two points in question. These equations give simple solutions in most cases. In dealing with the continuous beam and the restrained or built-in beam, the necessary formulas have been deduced by considering the effect of the loads acting separately. The sections dealing with torsion, spheres and cylinders under uniform pressure, and flat plates, contain the usual formulas for these cases.

The feature of the book that appealed most strongly to the reviewer was the

large number of well graded applications that have been included. The solving of these applications cannot fail to give the student a better understanding of the fundamentals involved and at the same time should tend to stimulate his interest in the subject.

WILLIAM A. JOHNSTON.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

PROBLEMS FOR SOLUTION.

ALGEBRA.

438. Proposed by WALTER C. EELLS, U. S. Naval Academy, Annapolis, Maryland.

In Hardy's *Pure Mathematics* (page 14, Nos. 2, 3) occurs the problem: "Show that if m/n is a good approximation to $\sqrt{2}$, then $(m+2n)/(m+n)$ is a better one, and that the errors in the two cases are in opposite directions, *e. g.*, 1/1, 3/2, 7/5, 17/12, 41/29, 99/70, $\cdots$." Find (a) other approximations for $\sqrt{2}$ of same type, *i. e.*,

$$\frac{m'}{n'}=\frac{am+bn}{cm+dn},\qquad (a,\ b,\ c,\ d,\ m,\ n,\ \text{integers}).$$

(b) Similar approximations for the square roots of other integers.

439. Proposed by A. M. KENYON, Purdue University.

If k, n are natural numbers, $n>2k$, show that

$$\frac{2^k}{\underline{|k}}\sum_{i=0}^{I\left(\frac{n+1}{2}\right)}\frac{1}{\underline{|2i+1}\,\underline{|n-k-2i}}=\frac{2^n}{\underline{|n+1}}\sum_{i=0}^{k}\binom{n-i}{n-k},$$

where $I(n/2)$ denotes the integral part of $n/2$ and $\binom{n}{k}$ is the coefficient of x^k in $(1+x)^n$.

440. Proposed by W. D. CAIRNS, Oberlin College.

n being a positive integer, find the sum of the series

$$n^2+4(n-1)^2+2(n-2)^2+4(n-3)^2+2(n-4)^2+\cdots,\tag{1}$$

where the succeeding coefficients are alternately 4 and 2; or, more generally, the series

$$an^2+b(n-1)^2+a(n-2)^2+b(n-2)^2+b(n-3)^2+\cdots.\tag{2}$$

L'Intermédiaire, July, 1913.

GEOMETRY.

469. Proposed by W. F. FLEMING, Chicago, Ill.

A pole whose length is l stands vertically against a vertical wall. A spider is at each end of the pole. The pole is drawn out from the wall in such a way that its upper end moves down the wall at a uniform rate. At the same time that the pole begins to move, the spiders begin to travel toward each other at rates equal to the rates at which the respective ends move. Determine the equations of the paths of the two spiders, in space.

470. Proposed by ROBERT E. MORITZ, University of Washington.

Prove that

$$\theta=\left(\lambda+\frac{q}{p}\mu\right)\pi,\qquad (\lambda=1,\ 2,\ 3,\ \cdots,\ q-1;\ \mu=0,\ 1,\ 2,\ \cdots,\ p-1)$$

and

$$\theta = (2\lambda - 1)\frac{\pi}{2} + \frac{q}{p}(4\mu \pm 1)\frac{\pi}{2}, \qquad \left(\lambda = 1, 2, 3, \cdots, \frac{q-1}{2};\ \mu = 0, 1, 2, \cdots, p-1\right);$$

determine the same set of points on the curve

$$\rho = a \cos \frac{p}{q}\theta,$$

where p and q are two odd integers without a common factor, and a is any constant.

471. Proposed by C. N. SCHMALL, New York City.

In the ellipse $x^2/a^2 + y^2/b^2 = 1$, an *equilateral* hexagon is inscribed with two sides parallel to the major axis. In the major auxiliary circle the same thing is done. If H_1 and H_2 be the sides of the hexagons, and e the eccentricity of the ellipse, show that $H_1 : H_2 :: 4 - 2e^2 : 4 - e^2$

CALCULUS.

390. Proposed by WILSON L. MISER, University of Minnesota.

Show that the triangle whose area is a constant and whose perimeter is a minimum is equilateral.

391. Proposed by H. B. PHILLIPS, Massachusetts Institute of Technology.

If $0 < \lambda < 1$ and $0 < x < \pi$, show that the function $(\sin \lambda x)/(\sin x)$ increases as x increases.

392. Proposed by HORACE OLSON, Student at The University of Chicago.

Two right circular cylinders of radii a and b respectively, are placed so that their axes intersect at right angles. Find the volume common to them.

MECHANICS.

314. Proposed by C. N. SCHMALL, New York City.

A rectangular box of height h, and having a plane mirror for its bottom, contains a quantity of water of unknown height x. In the lid are two small apertures distant $2a$ from each other. A ray of light entering one aperture with an angle of incidence ϕ, emerges, after refraction and reflection, through the other aperture. If μ be the index of refraction of water, show that the height of the water is

$$x = \frac{(h \tan \phi - a)}{\left[\tan \phi - \dfrac{\sin \phi}{(\mu^2 - \sin^2 \phi)^{\frac{1}{2}}}\right]}.$$

SOLUTIONS OF PROBLEMS.

ALGEBRA.

427. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

If $r \sin(\theta + \alpha) = m$, and $r \cos(\theta + \beta) = n$, show that

$$r = \frac{\sqrt{m^2 + n^2 - 2mn \sin(\alpha - \beta)}}{\cos(\alpha - \beta)}.$$

SOLUTION BY J. H. KELLOGG, Oberlin College.

From trigonometry, using only the positive square root, we know that

$$\sin(A+B) = \sqrt{(\sin A \cos B + \cos A \sin B)^2}.$$

$$= \sqrt{\sin^2 A(1-\sin^2 B) + (1-\sin^2 A)\sin^2 B + 2\sin A\cos A\sin B\cos B}$$

$$= \sqrt{\sin^2 A + \sin^2 B + 2\sin A\sin B(\cos A\cos B - \sin A\sin B)}$$

$$= \sqrt{\sin^2 A + \sin^2 B + 2\sin A\sin B\cos(A+B)}.$$

From the given equations we have

$$\sin(\theta+\alpha) = \frac{m}{r} \quad\text{and}\quad \sin\left[\frac{\pi}{2} - (\theta+\beta)\right] = \frac{n}{r}.$$

Hence letting

$$A = (\theta+\alpha) \quad\text{and}\quad B = \left[\frac{\pi}{2} - (\theta+\beta)\right],$$

we have

$$(A+B) = \left[\frac{\pi}{2} + (\alpha-\beta)\right].$$

Substituting, we have

$$\sin\left[\frac{\pi}{2} + (\alpha-\beta)\right]$$

$$= \sqrt{\sin^2(\theta+\alpha) + \sin^2\left[\frac{\pi}{2} - (\theta+\beta)\right] + 2\sin(\theta+\alpha)\sin\left[\frac{\pi}{2} - (\theta+\beta)\right]\cos\left[\frac{\pi}{2} + (\alpha-\beta\right]},$$

or

$$\cos(\alpha-\beta) = \sqrt{\frac{m^2}{r^2} + \frac{n^2}{r^2} - 2\frac{m}{r}\frac{n}{r}\sin(\alpha-\beta)}.$$

Solving for r, we have

$$r = \frac{\sqrt{m^2+n^2-2mn\sin(\alpha-\beta)}}{\cos(\alpha-\beta)}.$$

Also solved by B. J. Brown, Herbert N. Carlton, Nathan Altshiller, C. E. Horne, C. N. Schmall, Paul Capron, J. A. Caparo, Frank Irwin, Frank R. Morris, S. A. Joffe, V. M. Spunar, L. G. Weld, George W. Hartwell, Elijah Swift, A. W. Smith, Joseph B. Reynolds, Richard Morris, Walter C. Eells, Albert N. Nauer, Elmer Schuyler, and A. M. Harding.

428. Proposed by FRANK IRWIN, University of California.

If the roots of the equation

$$x^n - na_1x^{n-1} + \binom{n}{2}a_2x^{n-2} + \cdots = 0$$

are all real, the condition that they should all be equal is $a_1^2 = a_2$. A proof of the sufficiency of the condition is readily obtained from a consideration of derivatives. A proof is desired not based on such considerations.

Solution by Elijah Swift, University of Vermont.

We are to show that if the roots are all real, and if $a_1^2 = a_2$, the roots are all equal. For that purpose write the above equation in the form

$$(x - a_1)^n + \binom{n}{2}(a_2 - a_1^2)x^{n-2} + \varphi(x) = 0,$$

where $\varphi(x)$ is a polynomial of degree $n - 3$ at most. Suppose $a_2 = a_1^2$, and write $y = x - a_1$. The equation becomes

$$y^n + \varphi(y + a_1) \equiv y^n + c_3y^{n-3} + \cdots + c_n = 0.$$

If all the c's are zero, the theorem is true. If not, let c_k be the first coefficient not zero, and c_e the last. The equation is then

$$y^n + c_ky^{n-k} + \cdots + c_ey^{n-l} = 0.$$

Apply Descartes's rule to this equation. If we call the left hand side $f(y)$, $f(y)$ and $f(-y)$ can have together at most $n - k + 1 - l$ variations of sign (k odd), or $n - k + 2 - l$ (k even), hence at most $n - 2 - l$. The equation has, therefore, at least $2 + l$ zero or complex roots. Exactly l roots are zero, however, hence it must have 2 complex. This contradicts our hypothesis that the roots were all real. Hence all the c's are zero and $f(y) \equiv y^n$, and the equation in x is $(x - a_1)^n = 0$.

Also solved by LAENAS G. WELD.

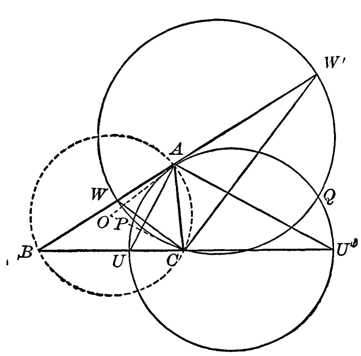
GEOMETRY.

456. Proposed by J. W. CLAWSON, Ursinus College.

The interior and exterior bisectors of the angles A, B, C of a triangle meet the opposite sides in U, U'; V, V'; W, W' respectively. Circles are drawn on UU', VV', WW' as diameters (Circles of Apollonius). Prove that (1) these three circles have a common chord. (2) The center of the circumcircle lies on this common chord.

SOLUTION BY PROPOSER.

1. $A(BC, UU')$ is a harmonic pencil; so (BC, UU') is a harmonic range. If T be any point on the circle having UU' as diameter, $T(BC, UU')$ is a harmonic



pencil. But UTU' is a right angle. Therefore TU bisects $\angle BTC$. Hence,

$$BT : TC = BU : UC = BA : AC.$$

In particular, if the circles having UU' and WW' as diameters intersect at the points P and Q,

$$BP : PC = BA : AC \text{ and } BQ : QC = BA : AC.$$

Similarly,

$$AP : PB = AC : CB \text{ and } AQ : QB = AC : CB.$$

Hence,

$$AP : PC = AB : BC \text{ and } AQ : QC = AB : BC.$$

It follows that P and Q are points on the circle having VV' as diameter. Hence the three circles UU', VV', WW' are coaxial.

2. Since (BC, UU') is a harmonic range, any circle passing through B and C is orthogonal to the circle having UU' as diameter. Hence, the circumcircle is orthogonal to that circle. Similarly the circumcircle is orthogonal to the circles VV' and WW'.

Hence, if O be the circumcenter, OA is a tangent to UU', OB to VV' and OC to WW'. But these tangents are equal in length.

Hence O is a point on the common chord of the circles UU', VV', WW'.

Also solved by C. E. HORNE, DAVID F. KELLEY, ELIJAH SWIFT, ROGER A. JOHNSON, and C. N. SCHMALL.

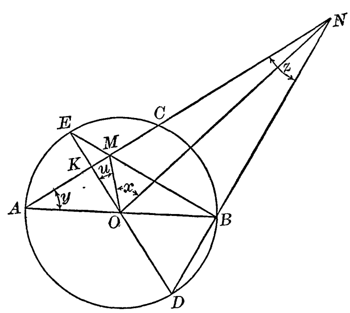
457. Proposed by NATHAN ALTSHILLER, University of Washington.

AB and AC are respectively a diameter and a chord of a circle whose center is O. The lines joining B to the extremities of the diameter perpendicular to AC, meet AC in the points M, N. Express the angle MON in terms of the angle CAB.

SOLUTION BY J. A. CAPARO, University of Notre Dame.

Let E and D be the extremities of the diameter perpendicular to AC at the point K. With the notation of the figure we easily see that

$$\sin \angle AMB = \sin \angle ABN = \cos z.$$



Hence, by the sine law, letting $AB = 2R$, we have in $\triangle ANB$, $AN = 2R \cot z$, and in $\triangle AMB$, $AM = 2R \tan z$. Also from $\triangle AKO$, $AK = R \cos y$, $OK = R \sin y$. $KN = AN - AK = 2R \cot z - R \cos y$, $MK = AM - AK = 2R \tan z - R \cos y$.

$$\therefore \tan u = \frac{MK}{OK} \frac{2 \tan z - \cos y}{\sin y}. \quad \text{Also } \tan (x + u) = \frac{KN}{KO}.$$

Hence, substituting the values of KN and OK,

$$\frac{\tan x + \tan u}{1 - \tan x \tan u} = \frac{2 \cot z - \cos y}{\sin y}.$$

Substituting the value of $\tan u$,

$$2 \sin y(\tan z - \cot z) = 2 \tan x \cos y(\cot z + \tan z) - 5 \tan x.$$

But $\tan z - \cot z = -2 \tan y$ and $\cot z + \tan z = \dfrac{2}{\cos y}$.

Hence, substituting and reducing,

$$\tan x = 4 \tan y \sin y,$$

the required relation.

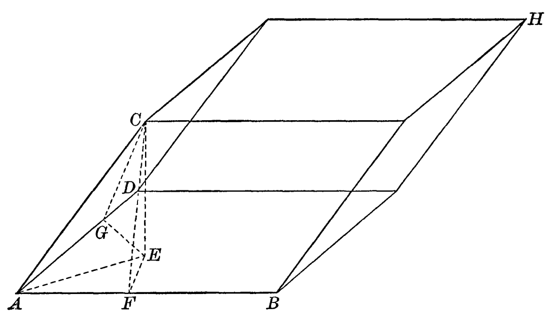
Also solved by PAUL CAPRON and J. W. CLAWSON.

458. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

Given edges l, m and n of a parallelopiped and angles a, b and c which the edges make with one another. Show that, if $s = \dfrac{a + b + c}{2}$, the volume equals

$$2lmn\sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}.$$

SOLUTION BY FRANK R. MORRIS, Glendale, Cal.



Given the parallelopiped AH with $\angle BAC = a$, $\angle DAC = b$, $\angle BAD = c$, and $AB = l$, $AC = m$, $AD = n$.

The area of the base BD is equal to $ln \sin C$.

From the vertex C drop a perpendicular to the base BD meeting it at E. From E draw perpendiculars to AB and AD meeting the lines at F and G, respectively. CE is the altitude of the parallelopiped and we know that the volume is $V = CE \cdot ln \sin c$. (1).

The triangles AEC, AFC, AGC, AFE and AGE are right triangles. Hence, we have

$$CE^2 = m^2 - AE^2, \quad (2) \qquad AF = m \cos a, \; AG = m \cos b, \quad (3)$$

$$\cos \angle EAF = \frac{AF}{AE}, \quad (4) \qquad \text{and} \qquad \cos (c - \angle EAF) = \frac{AG}{AE},$$

or

$$\cos c \cos \angle EAF + \sin c \sqrt{1 - \cos^2 \angle EAF} = \frac{AG}{AE}. \tag{5}$$

E lim inating $\cos \angle EAF$ from (4) and (5)

$$\cos c \frac{AF}{AE} + \sin c \sqrt{1 - \frac{AF^2}{AE^2}} = \frac{AG}{AE}.$$

From this equation we get

$$AE^2 = \frac{AG^2 - 2AG \cdot AF \cos c + AF^2}{\sin^2 c}.$$

Substituting the values of AF and AG from (3),

$$AE^2 = \frac{m^2 \cos^2 b - 2m^2 \cos a \cos b \cos c + m^2 \cos^2 a}{\sin^2 c}.$$

Then from (2),

$$\begin{aligned}
CE^2 &= \frac{m^2}{\sin^2 c}(\sin^2 c - \cos^2 b + 2 \cos a \cos b \cos c - \cos^2 a) \\
&= \frac{m^2}{\sin^2 c}(\sin^2 c + \cos^2 c - \cos^2 b - \cos^2 c + \cos^2 b \cos^2 c - \cos^2 b \cos^2 c \\
&\qquad + 2 \cos a \cos b \cos c - \cos^2 a) \\
&= \frac{m^2}{\sin^2 c}\{(1 - \cos^2 b - \cos^2 c + \cos^2 b \cos^2 c) - \cos^2 a + 2 \cos a \cos b \cos c - \cos^2 b \cos^2 c\} \\
&= \frac{m^2}{\sin^2 c}\{(1 - \cos^2 b)(1 - \cos^2 c) - \cos^2 a + 2 \cos a \cos b \cos c - \cos^2 b \cos^2 c\} \\
&= \frac{m^2}{\sin^2 c}\{\sin^2 b \sin^2 c - \cos^2 a + 2 \cos a \cos b \cos c - \cos^2 b \cos^2 c\} \\
&= \frac{m^2}{\sin^2 c}\{\sin b \sin c + (\cos a - \cos b \cos c)\}\{\sin b \sin c - (\cos a - \cos b \cos c)\} \\
&= \frac{m^2}{\sin^2 c}\{\cos a - \cos (b + c)\}\{\cos (b - c) - \cos a\} \\
&= \frac{m^2}{\sin^2 c}\left\{2 \sin\left(\frac{a+b+c}{2}\right) \sin\left(\frac{b+c-a}{2}\right)\right\}\left\{\sin\left(\frac{a-b+c}{2}\right) \sin\left(\frac{a+b-c}{2}\right)\right\} \\
&= \frac{4m^2}{\sin^2 c} \sin s \cdot \sin (s - a) \sin (s - b) \sin (s - c),
\end{aligned}$$

where

$$s = \frac{a+b+c}{2}.$$

Hence, $CE = \dfrac{2m}{\sin c}\sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}$. Substituting this value of CE in (1), we have

$$v = 2lmn\sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}.$$

Also solved by GEORGE W. HARTWELL, HORACE OLSON, J. A. CAPARO, and J. W. CLAWSON.

CALCULUS.

370. Proposed by PAUL CAPRON, Annapolis, Maryland.

The surface of a right circular cone having the semi-vertical angle α is cut by two planes, which intersect the axis at the same point, one at right angles to the axis, the other making the angle $(90° - \beta)$ with the axis. Show that if the lateral surface of the right cone is S_1 and that of the oblique cone S_2,

$$S_2 = \sum_1^\infty T_n, \qquad \text{where} \qquad T_1 = S_1,\ T_{n+1} = T_n \times \frac{2n+1}{2n}(\tan \alpha \tan \beta)^2.$$

SOLUTION BY THE PROPOSER.

Let the vertex of the cone be the origin, its axis the z-axis, and let the intersection of the oblique plane with the xy-plane be parallel to the y-axis. Let the two planes cut the z-axis at $(0, 0, h)$, and let the radius of the right section be a. Then,

$$\sqrt{x^2+y^2} = r, \text{ and } y/x = \tan\theta.$$

We have the following equations:

For the cone, $z = k \cot \alpha$, and for the oblique plane, $z = h - x \tan \beta x$. Eliminating z, we have for the projection on the xy-plane of the oblique section,

$$r = \frac{h \tan \alpha}{1 + \tan\alpha \tan\beta \cos\theta}.$$

If dA is an element of area in the xy-plane, the element of the conical surface of which it is the projection is $\csc \alpha \cdot dA$. Hence $dS_2 = \csc \alpha \, r d\theta dr$, and

$$S_2 = \tfrac{1}{2} h^2 \csc\alpha \tan^2\alpha \int_0^{2\pi} \frac{d\theta}{(1+\tan\alpha\tan\beta\cos\theta)^2}$$

$$= \frac{h^2}{2} \sec\alpha\tan\alpha \int_0^{2\pi} (1 + k\cos\theta)^{-2} d\theta,$$

where

$$k = \tan\alpha\tan\beta.$$

$$\int_0^{2\pi} (1+k\cos\theta)^{-2}d\theta = \int_0^{2\pi} (1 - 2k\cos\theta + 3k^2\cos^2\theta - \cdots)d\theta$$

$$= 2\pi\left[1 + \frac{3}{2}k^2 + \frac{15}{8}k^4 + \frac{35}{16}k^6 + \cdots\right],$$

where the general term is

$$T_{n+1} = (2n+1)k^{2n}\int_0^{2\pi} \cos^{2n}\theta d\theta = \frac{(2n+1)k^{2n}(2n-1)(2n-3)\cdots 1}{2n(2n-2)\cdots 2}\frac{\pi}{2} \times 4.$$

Hence,

$$T_{n+1} = T_n \times \frac{2n+1}{2n} k^2,$$

where

$$k = \tan\alpha\tan\beta.$$

$$T_1 = \pi k^2 \tan\alpha \sec\alpha = \pi a^2 \csc\alpha = S_1.$$

Also solved by Albert N. Nauer.

371. Proposed by B. F. FINKEL, Drury College.

Prove that the shortest distance between two curves or surfaces is normal to each.

Solution by Elijah Swift, University of Vermont.

Let us prove this for two surfaces; the proof for two curves is similar. Suppose the equations of the surfaces are $f(x, y, z) = 0$, $F(x, y, z) = 0$, and that the shortest distance is between the points (x_1, y_1, z_1) on f and (x_2, y_2, z_2) on F. Then we wish to make $(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$ a minimum with the auxiliary conditions $f(x_1, y_1, z_1) = 0$, $F(x_2, y_2, z_2) = 0$. Necessary conditions are that the partial derivatives of

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + \lambda f(x_1, y_1, z_1) + \mu F(x_2, y_2, z_2),$$

where λ and μ are constants, shall all vanish. This gives us six equations, of which these are types:

$$2(x_1 - x_2) + \lambda \frac{\partial f}{\partial x_1} = 0,$$

$$-2(x_1 - x_2) + \mu \frac{\partial F}{\partial x_2} = 0.$$

From these, we have at once

$$\frac{\partial f}{\partial x_1} : \frac{\partial f}{\partial y_1} : \frac{\partial f}{\partial z_1} = x_1 - x_2 : y_1 - y_2 : z_1 - z_2.$$

But $\partial f/\partial x_1$, $\partial f/\partial y$, $\partial f/\partial z_1$, are proportional to the direction cosines of the normal to $f(x_1, y_1, z_1) = 0$ at (x_1, y_1, z_1). And $x_1 - x_2$, $y_1 - y_2$, $z_1 - z_2$, are proportional to the direction cosines of the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2). Hence this line coincides with the normal. Similarly it is normal to $F(x_1, y_1, z_1) = 0$. Hence, the shortest distance must be perpendicular to both surfaces, *but not necessarily conversely.*

372. Proposed by V. M. SPUNAR, Chicago, Illinois.

Find the condition that the equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \left(1 + \frac{a^2}{x^2}\right) y = 0$$

should have one solution expressible in integral powers of x; and show that if this condition is satisfied, every other solution of the equation possesses a logarithmic infinity at the origin.

SOLUTION BY ALBERT N. NAUER, St. Louis, Mo.

Let $y = vx^{-\frac{1}{2}}$. Then finding the first and second derivatives of y with respect to x and substituting these values in the equation, we have

$$\frac{d^2v}{dx^2} - v = \left(\frac{a^2 - \frac{1}{4}}{x^2}\right) v.$$

This equation is of the form

$$\frac{d^2u}{dx^2} - \alpha^2 u = \frac{m(m+1)}{x^2} u,$$

where $m(m + 1) = a^2 - \frac{1}{4}$, or $m = -\frac{1}{2} \pm a$ and $\alpha = 1$. This equation has the three following known solutions:

$$(1) \quad u = x^{-m}\left(1 - \frac{1}{m - \frac{1}{2}}\frac{\alpha^2x^2}{2^2} + \frac{1}{(m - \frac{1}{2})(m - \frac{3}{2})}\frac{\alpha^4x^4}{2!\,2^4} - \frac{1}{(m - \frac{1}{2})(m - \frac{3}{2})(m - \frac{5}{2})}\frac{\alpha^6x^6}{3!\,2^6} + \cdots\right),$$

$$(2) \quad u = e^{\alpha x}x^{-m}\left(1 - \frac{m}{m}\alpha x + \frac{m(m-1)}{m(m - \frac{1}{2})}\frac{\alpha^2x^2}{2!} + \frac{m(m-1)(m-2)}{m(m - \frac{1}{2})(m-1)}\frac{\alpha^3x^3}{3!} - \cdots\right),$$

$$(3) \quad u = e^{-\alpha x}x^{-m}\left(1 + \frac{m}{m}\alpha x + \frac{m(m-1)}{m(m - \frac{1}{2})}\frac{\alpha^2x^2}{2!} - \frac{m(m-1)(m-2)}{m(m - \frac{1}{2})(m-1)}\frac{\alpha^3x^3}{3!} + \cdots\right).$$

Solution (2) is a finite primitive solution when m is an integer. When m is not an integer or zero, solution (1) gives an infinite series in powers of x. To have integral powers of x, m must equal $-(2n+1)/2$ when n is any integer or zero; also in the original equation $\pm a$ must equal $(2n+3)/2$.

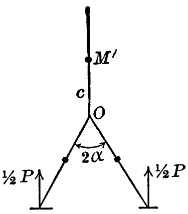
MECHANICS.

293. Proposed by B. F. FINKEL, Drury College.

A man of weight w stands on smooth ice; prove that if, when he gradually parts his legs, kept straight, with his feet in contact with the ice, the pressure of his feet on the ice be constant, his head will descend with uniform acceleration; and that, if f be the acceleration of his head, when his feet exert no pressure on the ice, their pressure on the ice, if f' were the acceleration of his head, would be equal to $\dfrac{f-f'}{f}w$. Walton's *Problems in Theoretical Mechanics*, p. 662.

SOLUTION BY E. B. WILSON, Massachusetts Institute of Technology.

We may analyze the man's total mass, M, into M'', the mass of the legs, and M', the remaining mass. The forces acting are W down and P up. Let 2α be the angle between the legs, l their length, $a\cos\alpha$ the distance of their center of gravity below O. Let c be the distance of the center of gravity of M' above O. The head falls through the distance $l(1-\cos\alpha)$. The center of gravity of the whole mass is at a height



$$h=\frac{M'(c+l\cos\alpha)+M''(l-a)\cos\alpha}{M=M'+M''}$$

above the ice. Hence the downward acceleration of the head is

$$f'=-l\frac{d^2\cos\alpha}{dt^2}.$$

The downward acceleration of his C.G. is

$$-\frac{M'l+M''(l-a)}{M}\frac{d^2\cos\alpha}{dt^2}=\frac{W-P}{M}.$$

Hence,

$$\frac{M'l+M''(l-a)}{M}\frac{f'}{l}=\frac{W-P}{M}.$$

If P is constant, then f', which is the only possible variable in this equation, must also be constant. This proves the first part.

If f be the value of f' when $P=0$, we have

$$\frac{M'l + M''(l-a)}{M}\frac{f}{l} = \frac{W}{M}.$$

Hence,

$$\frac{P}{M} = \frac{M'l + M''(l-a)}{M}\frac{(f-f')}{l}$$

and

$$\frac{P}{W} = \frac{f-f'}{f}.$$

This proves the second part.

294. Proposed by EMMA GIBSON, Student at Drury College.

A sphere, revolving about a diameter and not acted on by any extraneous force, expands symmetrically; prove that its vis viva varies inversely as its moment of inertia about its diameter.

SOLUTION BY E. B. WILSON, Massachusetts Institute of Technology.

The moment of momentum of the sphere is $I\omega$, where I is the moment of inertia and ω the angular velocity about the axis. This is constant as no external forces are acting. The kinetic energy is $\frac{1}{2}I\omega^2$ or $I^2\omega^2/2I$, which proves the proposition.

MECHANICS.

295. Proposed by B. F. FINKEL, Drury College.

A homogeneous hollow cylinder, whose inner radius is half of its outer radius, rolls without slipping down a plane inclined at an angle α to the horizon. Find its acceleration.

I. SOLUTION BY A. M. HARDING, University of Arkansas.

The external forces acting are W pounds at the center vertically downward, the reaction normal to the plane, and the friction up the plane.

Let R denote the resultant of the last two and let β denote the angle that its direction makes with the normal. Then the equation of motion of the mass center is

$$W\frac{d^2s}{dt^2} = Wg\sin\alpha - R\sin\beta. \tag{1}$$

If the length of the outer radius of the cylinder is a, then

$$I\frac{d^2\theta}{dt^2} = R\,x\,a\sin\beta = Ra\sin\beta,$$

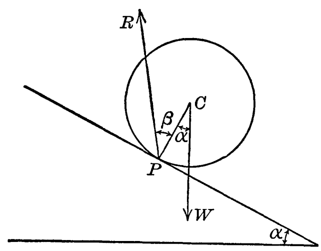
where the moment of inertia of the cylinder about its axis is $I = \frac{5}{8}Wa^2$.

But, since the cylinder does not slide,

$$s = a\theta. \quad \therefore \frac{d^2s}{dt^2} = a\frac{d^2\theta}{dt^2}.$$

Hence,

$$I\frac{d^2s}{dt^2} = Ra^2 \sin\beta, \qquad \text{or} \qquad R\sin\beta = \frac{I}{a^2}\cdot\frac{d^2s}{dt^2} = \frac{5W}{8}.$$



Substituting in equation (1), we obtain

$$\left(W + \frac{5}{8}W\right)\frac{d^2s}{dt^2} = Wg\sin\alpha. \quad \therefore \frac{d^2s}{dt^2} = \frac{8}{13}g\sin\alpha.$$

That is, 5/13 of the acceleration of gravity is used in turning the cylinder.

II. SOLUTION BY J. H. KELLOGG, Oberlin College.

Let M, I, R and r be the mass, moment of inertia, and outer and inner radii of the cylinder respectively; $P.E.$ and $K.E.$ the changes in potential and kinetic energy, due to the descent; v the linear velocity, ω the angular velocity, and a the acceleration, of a point on the outer circumference; d the length of the plane; and g the acceleration due to gravity.

Remembering that the $K.E.$ has two parts, one translational and the other rotational; and that

$$R = 2r, \qquad v = r\omega, \qquad v^2 = 2ad, \qquad \text{and} \qquad I = M\frac{R^2 + r^2}{2};$$

then

$$K.E. = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}Mv^2\left[1 + \frac{R^2 + \left(\frac{R}{2}\right)^2}{2R^2}\right] = \frac{13}{16}Mv^2 = \frac{13}{8}Mda,$$

and

$$P.E. = Mg\,(d\sin\alpha).$$

But the energy of the system must remain the same; hence

$$P.E. = K.E.$$

and

$$a = \frac{8}{13}g\sin\alpha.$$

NUMBER THEORY.

224. Proposed by PATRICK WALSH, New Orleans, Louisiana.

Find the sides, in rational numbers, of a right angled triangle whose area is $5\frac{1}{2}$.

SOLUTION BY C. E. HORNE, Westminster, Colorado.

Let $11x/y$ and y/x be the legs of the triangle, thus fulfilling one condition of the problem. Then

$$\frac{121x^2}{y^2}+\frac{y^2}{x^2}=\frac{121x^4+y^4}{x^2y^2}=\square.$$

This will be so when $121x^4+y^4=\square$. Let $\sqrt{121x^4+y^4}=11x^2+zy^2$. Then

$$x=y\sqrt{\frac{1-z^2}{22z}}$$

and

$$\frac{x}{y}=\sqrt{\frac{1-z^2}{22z}},$$

will be rational when

$$\sqrt{\frac{1-z^2}{22z}}=\square=\text{say } t^2.$$

Solving for z, we have $z=-11t^2\pm\sqrt{121t^4+1}$. When $t=3/70$ $z=-50/49$ or $49/50$. Hence, $x/y=3/70$; $11x/y=33/70$ and $y/x=70/3$. The hypotenuse $=4901/210$.

NOTE.—Having found one value of t which makes $\sqrt{121t^4+1}$, rational, we may find other values by making use of Euler's method, in his *Elements of Algebra*, third edition, revised by Hewlett, Chapter IX, p. 374.—EDITOR.

225. Proposed by W. DE W. CAIRNS, Oberlin College.

L'Intermédiaire for June, 1914, contains the following problem: "If we write the terms of the arithmetic series 1, 5, 9, 13, 17, 21, 25, 29, 33, $\cdots$ as follows:

1						
5	9	13				
17	21	25	29	33		
37	41	45	49	53	57	61
. . .	. . .	. . .	. . .	. . .	. . .	. . . ,

it is seen that the sum of the terms of each line is a cube, and that these are the cubes of the successive odd integers. How is this shown?"

It is here proposed not only to prove this, but to generalize the theorem as suggested, using, however, the simpler (and better known) case which includes all of the successive integers:

1			
3	5		
7	9	11	
13	15	17	19
. .	. .	. .	. . .

SOLUTION BY THOS. E. MASON, Purdue University.

Theorem. Form the arithmetic series $1+2kn$, $n=0, 1, 2, 3, \cdots$, where k is any positive integer which remains constant for a given series. Arrange this

series in rows, 1 in the first row, and each row after the first containing k more terms than the preceding. Then the sum of the terms in any row is the cube of the number of terms in that row.

From the way in which the series is formed, the value of n for any term will be the same as the number of terms preceding it in the series. The number of terms in the rows will form an arithmetic series with first term 1 and difference k. Making use of these facts, we obtain the following:

The number of terms in the rth row is $1 + (r - 1)k$.

The value of n for the first term in the rth row is the sum of $r - 1$ terms of the arithmetic series with first term 1 and difference k, that is,

$$\frac{r-1}{2}[2 + (r - 2)k].$$

Making use of this value of n, we have for the first term in the rth row the value $1 + k(r - 1)[2 + (r - 2)k]$. The sum of the terms in the rth row will be the sum of $1 + (r - 1)k$ terms of the arithmetic series with first term $1 + k(r - 1)[2 + (r - 2)k]$ and difference $2k$, that is,

$$\frac{1 + (r-1)k}{2}\{2[1 + k(r - 1)\{2 + (r - 2)k\}] + k(r - 1)2k\} = \{1 + (r - 1)k\}^3.$$

This proves the theorem.

The two arrangements of the problem can be obtained by making $k = 2$ and $k = 1$, respectively.

Also solved by R. M. Mathews, Elijah Swift, Harmon L. Slobin, and S. A. Joffe.

QUESTIONS AND DISCUSSIONS.

Edited by U. G. Mitchell, University of Kansas.

At the time of making up copy for this issue further replies are desired to questions numbered 4, 8, 12, 13, 16, 20, 23, 24, 25 and 26.

NEW QUESTIONS.

27. A certain college wishes to offer twelve hours of mathematics beyond the usual courses in analytical geometry and differential and integral calculus. Considering only the needs of students intending to specialize in pure mathematics, what courses should make up the twelve hours offered?

28. Is it possible to obtain $\int \cos \theta^2 d\theta$ without expanding $\int \cos \theta^2$? If it is not, can some interesting properties of this integral be determined by treating it as a special function?

DISCUSSIONS.

RELATING TO ADJUSTABLE CALENDARS.

By Irwin Roman, Chicago, Ill.

So far as the writer has been able to learn, all perpetual or adjustable calendars are arranged so as to present the first day of the month as the first day of the week.

This arrangement makes it necessary to look at the top of the calendar to see what day of the week a certain date is. The accompanying drawings illustrate a calendar which presents Sunday first as in the ordinary printed sheet calendar.

S	M	T	W	T	F	S

CUT OUT ALONG
HEAVY LINES

Apr. July	Jan. Oct.	May	Aug.	Feb. Mar. Nov.	Jun.	Sep. Dec.

PLACE MONTH OPPOSITE YEAR.

USE STARRED YEAR NUMBERS FOR JANUARY AND FEBRUARY OF LEAP YEARS.

						1	2	3	4	5	6	7
2	3	4	5	6	7	8	9	10	11	12	13	14
9	10	11	12	13	14	15	16	17	18	19	20	21
16	17	18	19	20	21	22	23	24	25	26	27	28
23	24	25	26	27	28	29	30	31				
30	31											

16	16 *	15	14	13	12	12 *	11	10	9	8	8 *	7	7 = 1907 8 = 1908
22	21	20	20 *	19	18	17	16	16 *	15	14	13	12	etc.

The figures are self-explanatory. To use the calendar, slide the top card over the bottom card until the name of the month and the number of the year desired are opposite each other. The calendar for that month will then appear

through the section cut from the top card. The details of construction and the reasons for the arrangements need not be given here, but will be furnished if requested. The period of years may be extended at will. The size and shape may also be varied to suit individual tastes.

RELATING TO SOLUTIONS OF QUADRATIC EQUATIONS.

By Geo. R. Dean, Missouri School of Mines.

I. *Solution of the Quadratic without factoring or completing the square.*

Let the equation be

$$ax^2 + bx + c = 0.$$

Put

$$x = u + iv, \qquad \text{where} \qquad i = \sqrt{-1}.$$

Then

$$a(u^2 - v^2 + 2uvi) + b(u + iv) + c = 0,$$

$$a(u^2 - v^2) + bu + c + i(2auv + bv) = 0,$$

Since the real and imaginary parts vanish separately,

$$a(u^2 - v^2) + bu + c = 0, \qquad \text{and} \qquad (2au + b)v = 0.$$

And since v is not, in general, equal to zero, we get

$$u = -\frac{b}{2a},$$

from which

$$au^2 + bu + c = c - \frac{b^2}{4a}.$$

Hence,

$$av^2 = c - \frac{b^2}{4a}, \qquad v = \frac{\pm\sqrt{4ac - b^2}}{2a};$$

$$u + iv = \frac{-b \pm i\sqrt{4ac - b^2}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

When v, that is, $\dfrac{\sqrt{4ac - b^2}}{2a}$ is imaginary the equation has real roots; and when $v = 0$, equal roots.

There is probably nothing new about this solution, but it affords a good example of the part played by the imaginary unit in higher mathematics, and would not be out of place in our elementary text-books on algebra.

II. *Solution of a Pair of Simultaneous Equations which occurs in the Theory of Cables and Transmission Lines.*

In the following equations the unknown quantities are α and β:

$$\alpha^2 - \beta^2 = RS - LCp^2 \quad (1); \qquad 2\alpha\beta = (RC + LS)p. \quad (2)$$

The solution by the regular algebraic process gives

$$\alpha = \sqrt{\tfrac{1}{2}\{\sqrt{(R^2 + p^2L^2)(S^2 + p^2C^2)} + (RS - LCp^2)\}}$$

$$\beta = \sqrt{\tfrac{1}{2}\{\sqrt{(R^2 + p^2L^2)(S^2 + p^2C^2)} - (RS - LCp^2)\}}.$$

A more elegant solution, from the mathematician's point of view, more convenient for the computer, and furnishing at the same time the value of three other quantities that are needed in other computations, is obtained by using trigonometric functions.

Let

$$\alpha = N \cos \xi,\ \beta = N \sin \xi, \text{ then } \alpha^2 - \beta^2 = N^2 \cos 2\xi;$$

$$R = Z \cos \delta,\ pL = Z \sin \delta, \text{ then } \tan \delta = \frac{pL}{R},\ Z = \frac{R}{\cos \delta} = \frac{pL}{\sin \delta};$$

$$S = Y \cos \gamma,\ pC = Y \sin \gamma, \text{ then } \tan \gamma = \frac{pC}{S},\ Y = \frac{S}{\cos \gamma} = \frac{pC}{\sin \gamma};$$

$$RS - LCp^2 = ZY \cos (\gamma + \delta),\ (RC + LS)p = ZY \sin (\gamma + \delta).$$

Therefore

$$N^2 \cos 2\xi = ZY \cos (\gamma + \delta),\ N^2 \sin 2\xi = ZY \sin (\gamma + \delta),$$

and it is easy to see that

$$N = \sqrt{ZY}, \text{ and } \xi = \tfrac{1}{2}(\gamma + \delta).$$

As a numerical illustration, take $R = 0.30$, $L = 0.00196$, $C = 0.0153 \times 10^{-6}$, $S = 0$, $p = 377$. Then, using a slide rule,

$$\tan \delta = \frac{pL}{R} = 2.4600,\ \delta = 67^\circ\ 53',\ \cos \delta = 0.3765,\ Z = 0.795;$$

$$\tan \gamma = \frac{pC}{S} = \infty,\ \gamma = 90^\circ,\ \xi = \tfrac{1}{2}(\gamma + \delta) = 78^\circ\ 56'\ 30'',\ Y = 5.77 \times 10^{-6};$$

$$\alpha \doteq \sqrt{ZY} \cos \xi = 0.000412,\ \beta = \sqrt{ZY} \sin \xi = 0.002100.$$

The quantities γ, δ, ξ are useful in other computations.

NOTES AND NEWS.

EDITED BY W. D. CAIRNS, Oberlin, Ohio.

Professor R. D. Carmichael has accepted a position in the University of Illinois.

Professor B. F. FINKEL was in charge of the mathematics at the summer school of the University of Colorado.

Mr. C. R. Dines, of Northwestern University, has been appointed to a position as instructor in mathematics at Dartmouth College.

Dr. H. B. PHILLIPS has been promoted to an assistant professorship of mathematics in the Massachusetts Institute of Technology.

Professor L. C. KARPINSKI, of the University of Michigan, and Mr. E. F. Gee of Detroit Central High School, were re-elected chairman and secretary, respec, tively, of the mathematics section of the Michigan Schoolmasters' Club at its April, 1915, meeting.

Mr. A. J. MILLER, instructor in mathematics at the University of Michigan, has been appointed by Harvard University to a travelling fellowship for the year 1915–1916. Mr. Miller expects to spend the year at Turin, Italy, studying with Professor Segre.

An article by Professor W. H. ROEVER of Washington University, on "The design and theory of a mechanism for illustrating certain systems of lines of force and stream lines," appeared in a recent number of *Zeitschrift für Mathematik und Physik.*

The death is reported of William B. Graves, professor emeritus of natural science at Phillips Academy, Andover, Mass., and for seven years professor of mathematics and civil engineering at the Massachusetts Agricultural College.

A work by G. F. HILL, of the British Museum, on the spread of the Hindu numerals in Europe has recently been published by the Clarendon Press, Oxford. A review will appear in a later issue.

Mr. J. C. NICHOLS, assistant professor of mathematics at the Texas Agricultural College, has been appointed to a fellowship in mathematics at the University of Michigan for the year 1915–1916. Mr. Nichols will devote himself to work in the history of mathematics.

Mr. CARL COE, who has been studying in Harvard University for the past two years, has resumed his position as instructor of mathematics in the University of Michigan.

An Italian edition of the work on "Mathematical Recreations and Problems," by W. W. ROUSE BALL, was published in 1911 (Bologna, N. Zanichelli), translated by Professor D. Gambioli.

The publishing house of A. Formiggini in Genoa has in progress a series of popular biographies (*Profili*), published at one lira, including in the numbers already issued *Archimedes*, by Favaro, and *Galileo Galilei*, by the same author.

Dr. Roger A. Johnson, of Western Reserve University, gave a course on "Geometry for Teachers" at the Harvard University summer school this summer.

On the list of those elected to membership in the American Philosophical Society occur the names of Professor W. F. OSGOOD of Harvard University, and Professor J. A. MILLER of Swarthmore College.

At Harvard University Dr. E. V. HUNTINGTON has been promoted to an associate professorship in mathematics, and at the University of Minnesota Dr. W. H. BUSSEY has been likewise honored.

Mr. C. H. YEATTON has been appointed to an instructorship in mathematics at Northwestern University. He has just completed his work for the doctorate at the University of Chicago.

Dr. H. S. WHITE, professor of mathematics in Vassar College, and Dr. R. A. MILLIKAN, professor of physics in the University of Chicago, were elected to membership in the National Academy of Sciences on April 21, 1915.

Professor E. F. CODDINGTON, of the department of mechanics in Ohio State University, will be acting dean of the college of engineering this year during the absence of Dean Orton; Professor C. C. MORRIS, of the department of mathematics, will act as assistant to the dean.

Professor A. B. COBLE of Johns Hopkins University, and Professor W. A. HURWITZ of Cornell University, have been chosen associate editors of the *Transactions of the American Mathematical Society.*

JOSEPH J. HARDY, professor of mathematics and astronomy, and for forty-five years a member of the faculty of Lafayette College, died on May 2, 1915.

Professor HENRY SUZALLO, of the department of education in Teachers College, Columbia University, and author of "The Teaching of Primary Arithmetic" and other educational texts, has been elected to the presidency of the University of Washington.

In *School Science and Mathematics* for June Professor M. O. TRIPP, of Olivet College, writes on "Some simple applications of elementary algebra to arithmetic," and Mr. H. C. WRIGHT, in a paper on "Mathematical equipment and its uses" reports some of the methods employed in the University of Chicago high school.

The January–February number of *Rendiconti del Circolo Matematico di Palermo* contains two articles by Americans: "Continuity of functions of infinitely many variables," by Professor W. D. A. WESTFALL, of the University of Missouri, and "Infinite developments and the composition property ($K_{12}B_1$) in general analysis," by Dr. E. W. CHITTENDEN, of the University of Illinois.

Mr. A. L. NELSON has been appointed to an instructorship in mathematics at the University of Michigan. Mr. Nelson is a graduate of the University of

Kansas and has just completed his work for the Doctorate at the University of Chicago.

Miss MARY E. WELLS has accepted a position as instructor in mathematics at Vassar College. During the past year she acted as supply at Oberlin College during the leave of absence of Dr. Mary E. Sinclair. Miss Wells has just taken the doctorate at the University of Chicago.

Dr. P. ZEEMAN, since 1902 professor of geometry and theoretical mechanics in the University of Leyden, died on May 8, 1915.

The *compte rendu* of the international conference on the teaching of mathematics held last April in Paris has now been published by the Georg press in Geneva, Switzerland, under the editorship of the secretary, Professor H. Fehr.

Mr. H. E. Webb, head of the department of mathematics in the Commercial and Manual Training High School in Newark, N. J., is chairman of a committee of the Society for the Promotion of Engineering Education, which has recently made a preliminary report on "Cooperation with secondary schools," which should be of interest to readers of the MONTHLY. It is printed in Volume XXII, 1915.

Dr. WILLIAM M. SMITH, of the University of Oregon, has been elected associate professor of mathematics at Lafayette College in place of the late Professor Hardy. Dr. Smith will also be registrar of the College.

Dr. MORGAN W. CROFTON, F.R.S., formerly professor of mathematics at Queen's College, Galway, and later professor of mathematics and mechanics at the Royal Military Academy, Woolwich, died on May 13, 1915, in his eighty-ninth year. He was the author of text-books and tracts on mechanics and also contributed papers to the leading mathematical journals. He wrote the chapter on mean values and probability in Williamson's "Integral Calculus," a topic in which he was a pioneer investigator.

The International Commission on the Teaching of Mathematics has recently issued, through the Bureau of Education at Washington, an important bulletin on the teaching of elementary and secondary mathematics in all the leading countries of the world. This bulletin, No. 45, 1914, was prepared by J. C. BROWN, and sets forth the nature of the mathematics taught in every school year, from the first through the twelfth, in the standard type of school. It should be in the hands of every teacher of mathematics, and may be secured by addressing the United States Commissioner of Education, Washington, D. C.

About eighty-five teachers of mathematics and allied sciences were present at the last meeting of the Michigan Schoolmasters' Club. The papers at this session were devoted entirely to practical phases of the teaching of high school mathematics. Correlation and real problems occupied a prominent place in the discussion. At the meeting for April 1916, it is proposed to continue the discussion of real problems in high school work, and also to discuss the relation of the various phases of higher mathematics to the elementary mathematics.

A new number of the *Encyclopédie des Sciences Mathématiques* was published recently. It is dated March 15, 1915, and is the first number of the French edition of this extensive work, which has appeared since the beginning of the present European War. It is an unusually large number, containing 292 pages, and is devoted to the principles of rational mechanics and to statistical mechanics. The article on the former of these two subjects is more than fifty per cent. larger than the corresponding article in the German edition, but the article on the latter subject is only slightly changed, except that a supplement is added bringing the work up to date.

In spite of war conditions, recent numbers of *L'Intermédiaire* have been sent to its subscribers. Although the character of problems proposed in this journal does not perhaps permit of their being offered in our department of Problems, it will be interesting to quote a few which are typical.

(1) Integrate the partial differential equation

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial x \partial y} = 0.$$

(2) Form a table of the possible and impossible values for h, less than an assigned number, given that

$$x^3 - hy^3 = z^2, \quad x^3 + hy^3 = t^2,$$

x, y, z, t and h being positive integers. For example, $x = 5$, $y = 1$, $z = 9$, $t = 13$, $h = 44$.

(3) Show that

$$a^2 \int_0^{\pi/2} \frac{\sin^4 \varphi \cos^2 \varphi \, d\varphi}{(a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)^4} - b^2 \int_0^{\pi/2} \frac{\sin^2 \varphi \cos^4 \varphi \, d\varphi}{(a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)^4} = 0$$

without calculating the definite integrals.

An interesting conference was held at the University of Indiana in May, 1915. It was the second annual conference on "Educational Measurements." There were fifteen or more speakers, and the program shows that a serious effort was made to discover some standards by which to judge educational work. The proceedings have been published in pamphlet form. The first conference, in 1914, resulted in trying out the Courtis Arithmetic Tests in grades V to VIII in twenty Indiana City school systems. The results were sent to the Psychological Laboratory of Indiana University, where they were studied and arranged in comparative tables and charts. The University has issued a pamphlet covering the results of this study under the title "Arithmetic: A Cooperative Study in Educational Measurements."

The eleventh annual session of the Association of Ohio Teachers of Mathematics and Science was held at the Ohio State University, April 2 and 3, under the presidency of Professor C. C. Morris of Ohio State University. Professor E. H. Taylor of the Eastern Illinois State Normal gave two lectures on "Recent tendencies in the teaching of secondary mathematics," in which among other things he recommended the establishment in this country of the "continuation school" now being conducted in Germany as a school intermediate in character between

the American trade schools and colleges of engineering. Mr. C. F. Geeting of the Canton high school read a paper on "The first two years in high school mathematics." The association directed the appointment of a large committee, representing all sections of the state, which is to draft a bill for introduction in the state legislature substituting the metric for the British system; this committee is also to conduct an educational campaign for its popular adoption and to urge the teaching of the metric system as early as the second and third school years.

Some interesting developments in connection with mathematical clubs among high school students have come to the attention of the MONTHLY. A club at Hutchinson, Kan., has discussed such topics as Euclid and his work, The trisection of an angle, The squaring of the circle, The use of graphs, The slide rule, History of our numerals, History of logarithms, Life of Pythagoras, with different proofs of his theorems, and a debate on the question whether one year of algebra and one year of geometry should be required for graduation from the high school, the decision resulting in the affirmative.

A club at the Wendell Phillips high school in Chicago has discussed the following topics: Pascal's theorem on a set of secant lines, The relation between the circumcenter of a triangle, the inradius, and the distance between these centers, The collinearity of the center of the nine point circle, the centroid, the orthocenter, and the circumcenter of a triangle.

A club at the Hyde Park high school, Chicago, has also accomplished some remarkable results, an account of which will soon be published in *School Science and Mathematics*, where also was published last year an extended report of a club at the Shattuck School, Faribault, Minn.

In all these clubs the membership and attendance are purely voluntary and the enthusiasm runs high.

L'Enseignement Mathématique, Vol. XVII, Nos. 1 and 2, Jan. 15 and Mar. 15, 1915, contain the following articles which are of particular interest to teachers of mathematics: (1) *The problem of interpolation and Taylor's formula*, by R. Suppantschitsch of Vienna, (2) *On the trinomial of the second degree*, $ax^2 + bx + c$, by P. Suchar, of Pau; (3) *On the teaching of mathematics*, by G. Fontené, of Paris; and (4) *Trigonometry and its relations to geometry*, by A. Streit in Bern. In the article by Streit the theorems of Ceva, Menelaus, and Ptolemy are derived from simple trigonometrical and geometrical considerations.

These numbers also give the questionnaire on an "Inquiry into the training of teachers of mathematics in secondary schools in different countries," as proposed by the Central Committee of the International Commission on the Teaching of Mathematics. "The Central Committee desires to continue its work, though renouncing the hope of summoning a conference. If the national subcommissions furnish the necessary documents, the work projected for 1915 will be collected in a pamphlet similar to that which was devoted to the conference at Paris." The parts of the questionnaire are devoted to (1) general preparation of candidates, (2) theoretical scientific teaching, (3) professional training, (4) subsequent improvement, (5) legal provisions as to teachers, (6) bibliography.

The *Bollettino di Bibliografía e Storia delle Scienze Matematiche,* a journal published under the editorship of Professor GINO LORIA, at Genoa, Italy, and devoted to the historical and bibliographical side of mathematics is now in its seventeenth year. The first three issues of 1914, the latest at hand, contain the following articles in Italian: (1) *Unpublished Contributions to the Correspondence of Evangelista Torricelli,* by Profesor Antonio Favaro; (2) *Concerning the Name, "Algoritmo,"* a discussion of the use of the word *Algorism* as a title and its recognition as part of the name of Mohammed ibn Musa al-Khowarizmi; and (3) *The Theory of Proportion in an Unpublished Manuscript of Evangelista Torricelli's,* by F. Podetti. This latter work connects directly with the attempts in the period of Galilei to explain in an easy way the substance of Euclid's definition of equal ratios and the rest of Book V, by means of the principles which Galilei wished to substitute for Euclid's definitions.

Each number contains, besides a leading article of the kind indicated bibliographical material topically distributed, as well as notes on current events of interest to mathematicians. Typographical errors in English, French and German quotations are fairly frequent. However, the journal certainly serves a useful purpose in stimulating an interest in the history of science, to which field Italians like COSSALI, LIBRI, LORIA, FAVARO and SCHIAPPARELLI have made most noteworthy contributions.

The following is from a letter of Professor Cajori written to the MONTHLY while he was traveling in Europe:

"I met in Zurich Professor F. Rudio, the editor-in-chief of the collected works of Leonhard Euler, which are now in course of publication. The war is causing some delay and other perplexing complications; nevertheless, the printing is progressing. I found Professor Rudio reading proof. He expressed appreciation of the contribution toward this enterprise of $1,000, made by the American Mathematical Society. I was disappointed when I learned that only about twenty subscriptions for the collected works came from the United States. Owing to lack of foresight in the management of our libraries, American mathematicians will not enjoy the convenience of easy access to Euler's works to a degree at all commensurate with the fine gift made by the American Mathematical Society. In view of the tremendous importance of this most prolific and genial eighteenth century mathematician, who figures in the early development of nearly every part of modern mathematics, the number of subscriptions from the United States should not be 20, but 220. The rapidly growing universities of the South, West and Middle West will find it later increasingly difficult and perhaps eventually impossible to secure the complete works of Euler. The edition will consist of 40 volumes. Subscriptions can still be secured through Professor F. Rudio (111 Dolderstrasse, Zurich) at the reduced rate of $5 per volume. As not more than three volumes are published per year, the yearly outlay is but slight."

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THE PROMOTION OF COLLEGIATE MATHEMATICS.

By H. E. SLAUGHT, University of Chicago.

At the meeting of the Chicago Section of the American Mathematical Society in April, 1914, there arose an informal discussion with respect to the field of collegiate mathematics. It was pointed out, on the one hand, that the interests of secondary mathematics have been well cared for in recent years both through the organization of effective secondary associations in all parts of the country, and through two representative secondary journals; and, on the other hand, that the interests of research in pure mathematics have been strongly fortified through the activities of the American Mathematical Society and its various publications, through the two research journals published under private university auspices, and through the multiplication of research fellowships at all the great universities.

In contrast to these two important phases of mathematical interests, it was pointed out that between them there is the great intermediate field of collegiate mathematics which so far has had no organized attention. No society is concerned particularly with this field and no journal represents its interests, except in so far as a few individuals have endeavored to do so in connection with the AMERICAN MATHEMATICAL MONTHLY. It was recognized that a large majority of the men and women in the mathematical faculties of the six hundred institutions of college grade in the United States are devoting themselves exclusively to teaching, and that very many even of those whose tastes and desires lie in the lines of research devote a good part of their time to teaching collegiate mathematics.

Furthermore, it appeared that mathematical students in our graduate schools come in large measure from smaller institutions in which the teachers are not members of the American Mathematical Society and have no professional contact with their colleagues in other institutions either by way of scientific investigation of through interchanging ideas on the many important questions still outstanding with respect to the collegiate curriculum in mathematics.

And, finally, it appeared that very many even within the present Society, and

certainly many outside, who might be desirous of engaging in mathematical activities, either in the beginnings of research or by contributing to the betterment of teaching, find themselves practically debarred from all opportunity either because the rungs of the research ladder are all placed high above their reach, with no gradual approaches, or because there is at present no forum in which discussions of collegiate mathematics are welcome.

This informal discussion, which took place at the Chicago Section dinner, led to the appointment, at the later business meeting, of a committee of the Section to consider whether it might be desirable to request the Council of the Society to deliberate upon the questions thus raised. This committee reported at the December, 1914, meeting of the Chicago Section, and upon its recommendation a resolution was passed and transmitted to the Council requesting the Council to consider the feasibility of conducting, under the auspices of the American Mathematical Society, a journal for the field now covered by the MONTHLY.

Upon the presentation of this resolution at the meeting of the Council a committee of five was appointed to report to the Council. The report of this Committee was discussed at length by the Council at the unusually large and representative meeting in New York in April, 1915, and finally the following resolution was passed with only two or three dissenting votes:

"It is deemed unwise for the American Mathematical Society to enter into the activities of the special field now covered by the AMERICAN MATHEMATICAL MONTHLY; but the Council desires to express its realization of the importance of the work in this field and its value to mathematical science, and to say that should an organization be formed to deal specifically with this work, the Society would entertain toward such an organization only feelings of hearty good will and encouragement."

While some members of the Council committee and some others in the Society feel that the Society might well broaden its scope of activity along the lines suggested and thus maintain its sphere of influence throughout the entire mathematical field, yet the decision of the Council was so emphatic as to leave no room for doubt concerning the present policy of the Society, both as to its own attitude toward the field of activity in question and as to the desirability of having this field provided for by an organization formed to deal specifically with this work.

Accordingly, in order to ascertain how large a number of persons engaged in teaching collegiate mathematics, or otherwise related to this field, would be sufficiently interested in forming such a new society to sign their names to a call for an organization meeting, a statement was sent out in June, 1915, explaining the situation and enclosing a reply postal. Approximately 350 replies have been received, of which only about six or seven express some form of disapproval. The communication doubtless reached some institutions after the beginning of the summer vacation and hence it is possible that some additional replies may be received this fall:

It is proposed to send out the notice early in December with the names of all signers received up to that time. The meeting will be called at Columbus, Ohio, in connection with the holiday convocation of the American Association for the Advancement of Science. The name of the new society, its precise character and policy, its relation to the AMERICAN MATHEMATICAL MONTHLY, etc., will be questions for full discussion and determination at the organization meeting.

It should be clearly understood that this whole movement is a matter of public concern, and is in no sense a private undertaking; nor is it an effort on the part of those interested in the MONTHLY to rescue it from impending bankruptcy. The MONTHLY is in sound financial condition and is seeking no rescue measures. Its friends and supporters are interested in this new movement for the same reasons which actuate the rest of the signers to the call for the organization meeting; namely, a sincere desire to promote the course of mathematics in this country in all its many and varied aspects, and especially in that field that has been so greatly neglected,—the field of collegiate mathematics.

HISTORY OF ZENO'S ARGUMENTS ON MOTION:

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

By FLORIAN CAJORI, Colorado College.

IX.

E. POST-CANTORIAN DISSENSIONS.

Very frank and brilliant in its mode of exposition but perhaps lacking the originality and depth of the writings of Dedekind and Georg Cantor is the *Allgemeine Functionentheorie* of Paul du Bois-Reymond, Tübingen, 1882. This book, which is contemporaneous with Cantor's creation, discusses the philosophy and theory of the fundamental concepts of quantity, limit, argument, function. He declares that the difficulties surrounding the idea of a limit are not mathematical in character but have their roots in the "simplest parts of our thinking, our conceptions or images" (Vorstellungen). He says that there are two conceptions, those of the idealist and those of the empiricist, "which have equal right to count as fundamental views of rigorous science,"[1] for neither yields contradictory results, at least in pure mathematics. The author presents both sides "with equal rigor," and does not award victory to either. The idealist defends the existence not only of what can be imagined, but also of things unimaginable;[2] he assumes a transcendental attitude. Accordingly he assumes the termination of series, such as those given by endless decimal fractions, which are really "given"

[1] Paul du Bois-Reymond, *Die Allg. Functionentheorie*, Tübingen, 1882, p. 2.

[2] P. du Bois-Reymond, *op. cit.*, pp. 110, 111.

only to a certain term; he recognizes the limit of these decimals. The empiricist says on the other hand, "everything that is scientifically established springs from sense-perception; whatever is unimaginable must be rejected; we must proceed from images to images (von Vorstellungen zu Vorstellungen);[1] he denies the existence of a limit of an endless decimal fraction and is satisfied to take into account the members of the decimal as far as you may wish. A continuum such as that of G. Cantor and Dedekind does not exist, because it cannot be imagined. Take the endless decimal $0.\alpha_1\alpha_2\alpha_3\cdots$,

"In a drawing . . . imagine the ends of the distances $0.\alpha_1$, $0.\alpha_1\alpha_2$, $\cdots$ marked off from the zero point and designated by fine, short lines drawn across the line segment. These cross lines become more and more dense, and they must sometime stop, because the means of drawing fail us and, at any rate, one cannot draw an unlimited number of marks. There follows now after a short interval following the mark $0.\alpha_1\alpha_2\cdots\alpha_p$, which we shall call the fog-interval, a fine mark, somewhat longer than the previous ones, which represents the limit. In the fog-interval before the limit all sorts of philosophic apparitions play their pranks. Here play the celebrated sophisms; it is this that the idealist resolves into the *quantitates infinitesimas*. The empiricist's drawing looks a little different. The marks are continued, even if the means of drawing do not permit the marks to remain distinct. Then the marks run into each other, to show that their width is greater than the distance of $0.\alpha_1\alpha_2\cdots\alpha_p$ from the limit. The drawing of further marks is stopped when a further advance of the marks can no longer be recognized and new marks would be drawn through the same place. That is then the limit."[2]

The process of a variable continually approaching but never reaching its limit is characterized by du Bois-Reymond as follows:[3]

"Between the conception of the idealist, who lets dx be infinitely small and connects it with the idea of something at rest, unchanging, and my conception which assumes dx finite and sufficiently small but likewise at rest, there slips in a third conception (mentioned page 83, 84) in which, as is commonly expressed, dx is a quantity in the act of disappearing (quantité évanouissante), hence a quantity which is continually varying toward zero. The idea of a quantity in continual flux is repugnant to me. It goes against my grain to have symbols in my formulas for quantities which set themselves in motion as soon as I look at the formulas, and hasten toward zero, which, however, they are permitted to reach only at the end of the computation. As long as the book is closed, there reigns profound silence. No sooner do I open it than there begins that race toward zero of all quantities affected by a d."

Feelings of this sort have doubtless come to many mathematicians of the older school. This perpetual approach without ever reaching the goal is wearisome. Why not divorce the variable and its limit from the limitations of time? Or if, as in mechanical problems, the idea of time seems difficult to eliminate, hasten the successive steps of approach to the limit, the times for these steps diminishing at a sufficiently rapid rate, so that the time elements form together a converging series. Thereby the limit is reached in a finite time. The "Achilles" is a concrete illustration of such approach and reach of the limit.

Eight years after the issue of du Bois-Reymond's book there appeared a posthumous monograph prepared by a privat-docent in philosophy at the University of Strassburg, Dr. Benno Kerry. It is a philosophic study of the theory of limits.[4] Familiar with the book of du Bois-Reymond, as well as with

[1] P. du Bois-Reymond, *op. cit.*, p. 149.
[2] P. du Bois-Reymond, *op. cit.*, pp. 122, 123.
[3] P. du Bois-Reymond, *op. cit.*, pp. 140, 141.
[4] Dr. Benno Kerry, *System einer Theorie der Grenzbegriffe. Ein Beitrag zur Erkenntnisstheorie.* Herausgegeben v. Gustav Kohn, Leipzig u. Wien, 1890.

the work of Dedekind and G. Cantor, Kerry produced a very illuminating publication. Neither Kerry nor du Bois-Reymond discuss Zeno directly.

In 1885 a discussion of Zeno's arguments against motion was opened up in France, which lasted over a decade and in which a large number of writers participated. At no other decade and in no other country was the discussion of the topic so persistent and general. Never before have philosophical writers based their solutions of Zeno's puzzles so persistently upon the postulate of the discontinuity of time and space. Among the participants in the discussion were the mathematicians Paul Tannery, G. Milhaud, G. Frontera, G. Mouret, and L. Couturat, but the leading part was taken by the French philosophers Ch. Renouvier, F. Evellin, G. Noël, V. Brochard, and G. Lechalas.

In 1885 the noted historian of mathematics, Paul Tannery, published an article, *Le concept scientifique du continu. Zênon d'Elée et Georg Cantor*,[1] which we noted at the beginning of this history as advancing a brilliant and novel explanation of the *purpose* of Zeno's arguments. It concluded with an account of Cantor's continuum. This article was supported by a paper from the pen of Gaston Samuel Milhaud, who was successively professor of mathematics in the lycées at Nizza, Havre, Lille and Montpellier and who has been, since 1895, professor of philosophy at the University of Montpellier. Otherwise Tannery's article received little attention. The discussions carried on during the decade do not center in Cantor's continuum. Cantor's ideas were then very recent and not yet fully elaborated. Milhaud's article above referred to bore the title, *Le concept du nombre chez les Pythagoriciens et les Eléates.* Before this Milhaud had written a philosophical dialogue on *La notion de limite en mathématiques*,[2] which avowedly aimed to throw light on the "Achilles." *A* and *B* discuss the subject. *A* declares that the mind cannot encompass at once an unlimited series of elements; *B* declares that this negative quality of our mind has nothing to do with the existence or non-existence of an attainable limit. Whether the limit is actually reached or not is of no interest to the mathematician and is foreign to mathematics. Though a clever dialogue, it was criticized by Ernest Jean Georges Mouret, ingénieur-en-chef de ponts et chaussées, for dodging the real issue in the "Achilles" and failure to remove the underlying difficulties.[3]

Victor Brochard wrote on Zeno in 1885, in a memoir to l'Académie des Sciences morales et politiques and again in 1893, in a paper entitled, *Les prêtendus sophismes de Zênon d'Élée.*[4] He is mainly concerned with the question of Zeno's purpose and favors traditional views rather than the views of Tannery and Milhaud. In the earlier publication Brochard expressed the belief that Zeno meant actually to deny all motion. In the later article he prefers to speak of Zeno's *arguments* rather than Zeno's *sophisms.*

[1] *Revue philosophique de la France et de l'étranger.* Dixième année, XX, 1885, Paris, pp. 385–410. Excepting the part on G. Cantor, this article is reproduced almost verbatim in P. Tannery's *Pour l'histoire de la science hellène*, Paris, 1887, Chapt. X, pp. 247–261.

[2] *Revue philosophique* [Th. Ribot], Vol. 32, Paris, 1891, p. 1.

[3] "Le problème d'Achille" by George Mouret, in *Revue philosophique*, Vol. 33, Paris, 1892, p. 67.

[4] *Revue de métaphysique et de morale*, I, 1893, Paris, pp. 209–215.

A pamphlet, entitled, *Étude sur les arguments de Zênon d'Élée contre le mouvement*, Paris, 1891, was published by G. Frontera, in the preface of which he states that he was induced to prepare this study by the fact that some young students of philosophy informed him that in certain lectures on philosophy which they had attended it was seriously taught that the "Achilles" could not be explained and that motion was an illusion of the senses. Frontera holds the views of a mathematician untouched by the ideas of G. Cantor. He speaks in the "Dichotomy" of an infinite *series* of parts, instead of an infinite *number* of parts, "parce que l'idée de nombre exclut l'idée de l'infini." He says that Zeno considered only *space*, while the subject calls for the consideration of three things, *space, time, velocity*. The sum of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc., of a distance cannot be infinite as Zeno claimed, but must be finite. In dealing with series, Frontera reveals no knowledge of St. Vincent and other writers on the "Achilles," except the contemporary French writers, V. Brochard, F. Evellin. He speaks in high terms of Evellin's work, *Infini et quantité*. He writes down the convergent series for the time required to overtake the tortoise. He rightly insists that the velocities of Achilles and the tortoise must be maintained to the end, that Zeno does not do this and arrives at an absurd conclusion by the tacit assumption of chimerical conditions. The "Stade" he considers simply a question of relative motion. G. Mouret[1] criticizes Frontera for passing from the series to its limit without perceiving that it is precisely at this point where the difficulty lies. Mouret declares that the "Achilles" constitutes really a criticism of the foundations of convergent series and of the infinitesimal calculus. Frontera made reply,[2] emphasizing his former contention that the difficulty with the "Achilles" has been the chimerical hypothesis introduced by Zeno and his followers for now nearly 2,500 years.

Exception to Mouret's remark that Zeno's arguments were a criticism of the fundamental principles of the calculus and of series was taken by Louis Couturat.[3] Mouret's claim that Zeno arbitrarily excludes the point of meeting in the "Achilles" is historically not a correct interpretation; Zeno claimed that, to arrive at a point of meeting, Achilles had to run through an infinity of spaces, which Zeno thought impossible. The nerve of Zeno's argument consists, according to Couturat, in the axiom which even to-day is accepted as evident by many good heads: "The actual infinite cannot be realized"; any movement must contain an actual infinity of parts, which Zeno declared to be impossible. Zeno's sophism, if it exists, is more subtle and more profound than Mouret realizes; it is not a gross paralogism the falsity of which at once leaps into view.

In the year 1893 the discussion continued unabated. Georges Noël contributed an article, *Le mouvement et les arguments de Zênon d'Élée*,[4] which in many respects is an able discussion, but the author unfortunately does not always

[1] *Revue philosophique*, Vol. 33, 1892, p. 67.
[2] *Revue philosophique*, Vol. 33, 1892, p. 311.
[3] *Revue philosophique*, Vol. 33, 1892, p. 314.
[4] *Revue de métaphysique et de morale*, I, 1893, pp. 107–125.

distinguish between what is assumed and what is the result of clear logic, with that sharpness that is necessary to clear up such a difficult subject as motion; the article is marred by hidden assumptions. Noël claims that as soon as one grasps the true nature of movement, the refutation of Zeno is easy. Zeno divided the displacement into an infinitely increasing number of smaller displacements, standing out as distinct events which must concur in producing the final event. These smaller displacements, infinite in number, are not all given; hence Zeno concludes that the final event cannot take place. But, says Noël, the smaller displacements are not at all veritable conditions for the occurrence of the final event. They are coördinate events, but not subordinate. The final event derives its raison d'être direct from the state of the motion and its velocity. Thereby all positions taken by the moving point are given and all are on an equal footing. Their order of succession in time is in no way an order of logical dependence. Logically they are all given with the motion. Little does it matter if they are infinite in number. Neither the moving body, nor the mind contemplating it is really bound to number them. They do not introduce any real divisions in the motion. It is one motion, and its continuity excludes all actual division. The particular motions do not exist, except from an arbitrary subjective view point. Thus the "Dichotomy" and "Achilles" are real sophisms, but of such a nature that the human mind is almost inevitably carried away by them, as long as it neglects to subject the fundamental principles to a critical analysis. Motion is not a succession of positions, it is a "becoming."

Noël attacks the advocates of discontinuity, such as Evellin, who postulate the existence of a minimum distance and minimum time. Noël argues that such minimum existence is disproved by the "Arrow" and the "Stade."

Many years before this, Ch. Renouvier[1] and F. Evellin[2] argued against the actual infinite and in favor of the discontinuity of space and time; the number of parts in which the path AB can be divided is either finite or infinite, but the number cannot be infinite without the inexhaustible being found exhausted, hence the number is finite. Besides his book on *Infini et quantité*, Evellin published articles in 1893[3] and 1894[4] in which he elaborates his ideas with reference to our topic. Noël's explanation of the nature of motion does not appeal to him; Noël's hypothesis of motion as a "becoming" affords only a moment's illusion of having escaped the difficulties set by the dialectician of Elea. Evellin declares that Zeno's four arguments present two branches of a dilemma. The "Dichotomy" and "Achilles" are aimed at the mathematicians with their infinity, the "Arrow" and "Stade" are aimed at the partisans of limited division. Making negation of the infinite, he explains Zeno by assuming the discontinuity of time and space and the existence of indivisible parts of space and

[1] *Esquisse d'une classification systematique des doctrines philosophiques*, T. I, Paris, 1885. Renouvier's proof of the non-existence of the actual infinite, as well as other proofs of this are critically examined by Georg Cantor in his article, "Mitteilungen zur Lehre vom Transfiniten" in *Zeitschr. f. Philosophie und philosophische Kritik*, Bd. 91. Separatabdruck, Halle, p. 22.

[2] François Evellin, *Infini et quantité*, 1880.

[3] "Encore a propos de Zénon d'Élée," *Revue de métaphysique et de morale*, I, p. 382.

[4] "La divisibilité dans la grandeur," *Revue de métaphysique et de morale*, II, pp. 129–152.

time. On the hypothesis of unlimited divisibility he claims that Zeno successfully disproved motion. The concept of a line, says Evellin, must be such as to explain motion. The line must have parts; motion exhausts the line and therefore also its parts; hence, those parts have a number, but this number escapes us. Each moment of motion must mark an advance, but there is no advance except in the exhaustible and the finite. The "Dichotomy" and "Achilles" make this very plain. Evellin denies that the "Stade" disproves the possibility of indivisibles.

$$\begin{array}{ccccccccc} a & b & c & \qquad & & a & b & c & \\ a_1 & b_1 & c_1 & \qquad & a_1 & b_1 & c_1 & & \\ a_2 & b_2 & c_2 & \qquad & & & a_2 & b_2 & c_2 \end{array}$$

If in an indivisible moment the indivisible lines represented by a, b, c, a_1, etc., shift, so that a_2 comes to c_1, the question arises, how can a_2 and b_1 have found time to meet each other? They must indeed pass each other, if the discontinuity of space is assumed, but they need not meet at all in case of discontinuity.

Another champion of discontinuity is Georges Lechalas. In 1895 he brought out a book, an *Étude sur l'espace et le temps*, Paris, of which an enlarged edition appeared in 1910. In 1893 he contributed a journal article on Zeno.[1] He proceeds on the maxim that, so far as the realized number of things in the actual world is concerned, number always means *finite* number. He distinguishes abstract from realized number. He made a study of G. Cantor and offers no objection to actual infinity, provided this concept is confined to abstract number. He claims contradiction in its realization. All realized aggregates, he says, can be counted by the ordinary process. He denies the possibility of a real continuum and hence concludes that both space and time are discontinuous. While agreeing with Evellin on matters of discontinuity, Lechalas disagrees with him on the existence of a minimum distance and a minimum time. Evellin's mode of escape from the clutches of the "Stade" does not satisfy him. If the atom is indivisible, says Lechalas, this is not a minimum of extension; extension is not a property of atoms, but a relation between them. A point in moving from one position to another occupies only a finite number of intermediate positions; hence, in the "Stade," a_2 and b_1 may not be in the same vertical line at any time. Using discontinuity as a magic wand, he gets a_2 to c_1 without meeting b_1.

[To be concluded in the November issue.]

[1] "Note sur les arguments de Zénon d'Élée" in *Revue de métaphysique et de morale*, I, 1893, p. 396.

THE THEORY OF POLES AND POLARS.

By M. B. PORTER, University of Texas.

In the text-books on analytic geometry the symmetry of the equation

$$\Sigma a_{ij}x_ix_j' = 0 \quad [P]$$

is used to show that [P] is the equation of the chord (plane) of contact of the point x' with regard to the conic (quadric)

$$Q = \Sigma a_{ij}x_ix_j = 0,$$

while the theorems relating to secant lines (planes) are usually deferred until projective properties are taken up—usually in a separate course.

In addition to necessitating a preliminary treatment of tangent lines and planes, this method has the inconvenience that it does not give the properties which make ruled constructions of the tangents possible.

It is the purpose of this note to show that the various geometric properties given by the vanishing of the bilinear covariant,

$$\Sigma a_{ij}x_ix_j' = 0,$$

can be deduced in an elementary and simple manner from the equation itself, and in a way that is applicable without modification to quadratic loci in any number of dimensions.

The treatment here given is for quadrics in three dimensions and the coordinates used are point coördinates, though the processes admit, of course, the usual dual interpretation.

1. Writing our quadric in the form

$$\begin{aligned} Q &= \sum_1^4 a_{ij}x_ix_j \\ &\equiv (a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4)x_1 + (a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4)x_2 \\ &\quad + (a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4)x_3 + (a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4)x_4 \\ &\equiv A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 = 0, \end{aligned}$$

where $a_{ij} = a_{ji}$ and $A_i = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4}x_4$. Thus $\Sigma a_{ij}x_ix_j' = 0$ is obtained by priming either the x's inside or the x's outside the parentheses.

A *vertex* is a set of x's which satisfy the four equations

$$A_i = 0,\ i = 1, 2, 3, 4, \tag{1}$$

and the quadric is non-singular if it has no vertex (system (1) of rank 4), a *cone* if it has one vertex (system (1) of rank 3), a *plane-pair* if it has a line of vertices (system (1) of rank 2), a double plane if it has a plane of vertices (system (1) of

rank 1). From this we have that P *vanishes identically if x' is a vertex, and passes through all the vertices if x' is not a vertex.*

2. Consider now the pencil of quadrics

$$\overline{Q} = k_1Q + k_2Q_1 = 0, \tag{2}$$

where

where $Q_1 = \Sigma b_{ij}x_ix_j$, and the polar of $\overline{Q}$ with respect to x' is

$$k_1\Sigma a_{ij}x_ix_j' + k_2\Sigma b_{ij}x_ix_j' = 0, \tag{3}$$

which is the equation of a pencil of planes. If $\overline{Q} = 0$ is a singular quadric of the pencil (2) and x' is one of its vertices, [3] vanishes identically, so that

$$P \equiv k_1\Sigma a_{ij}x_ix_j' \equiv - k_2\Sigma b_{ij}x_ix_j' = 0.$$

Theorem: *thus we have that the polars of x' with respect to all the quadrics of pencil* (2) *are the same, if x' is a vertex of a quadric of the pencil. Moreover, P will in this case pass through the vertices of all other singular quadrics of the pencil.*

3. The application of this theorem leads at once to the geometric properties in question.

First let Q_1 be two planes, Q_1 is then a singular quadric of the pencil with a line of vertices which we shall suppose passes through the point x'. The polar of x' thus passes through the vertices of all the *other* singular quadrics of the pencil, *i. e.*, through the vertices of the two cones passing through the two conics cut out of $Q = 0$ by $Q_1 = 0$. If now one of the planes of which $Q_1 = 0$ consists becomes tangent to $Q = 0$ the vertex of one of these cones will become the point of contact of this tangent plane, so that $P = 0$ is the equation of the plane of contact of x'. If now the point x' approach the surface of $Q = 0$ as limit, $P = 0$ approaches the tangent plane so that if x' is on $Q = 0$, $P = 0$ is the tangent plane at x'.

Furthermore, if x' is any point on the line of vertices of $Q_1 = 0$,

$$P = \Sigma a_{ij}x_i(k_1x_j' + k_2x_j'') = 0$$

will determine a pencil whose axis,

$$\Sigma a_{ij}x_ix_j' = 0, \Sigma a_{ij}x_ix_j'' = 0,$$

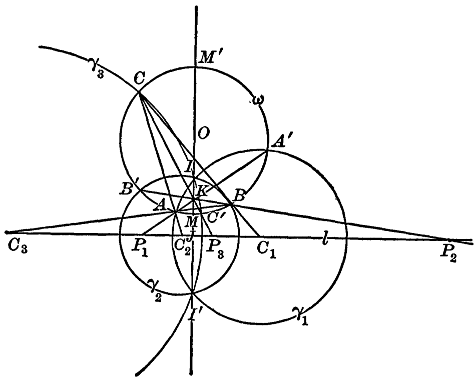
is the conjugate line of $(x'x'')$.

Secondly, if Q_1 is a double plane, its pole lies in P. More generally if $Q_1 = 0$ is any cone with vertex at x', $P = 0$ passes through the vertices of the three remaining cones which pass through the twisted quartic in which $Q = 0$ and $Q_1 = 0$ intersect. The existence of the remaining cones is shown, of course, by setting the discriminant of $\overline{Q}$ equal to zero.

ON THE CIRCLES OF APOLLONIUS.

By NATHAN ALTSHILLER, University of Colorado.

DEFINITION. The interior and exterior bisectors of the angles A, B, C of a triangle ABC meet the opposite sides in the points U, U'; V, V'; W, W' respectively. The circles described on the segments UU', VV', WW' as diameters, are called the *circles of Apollonius.*



Not all the lines considered are drawn in the figure.

(1) The bisectors CW, CW' being perpendicular to each other, the segment WW' subtends a right angle at the vertex C, therefore C is on the circle γ_3 described on WW' as diameter. Similarly for the points A and B with respect to the circles γ_1 and γ_2 described on UU' and VV' as diameters, and therefore:

The Apollonian circles pass through the respective vertices of the triangle.

(2) The bisectors CW, CW' separate harmonically the lines CA, CB,[1] hence any point S of the circle γ_3 is the center of the harmonic pencil of lines $S(ABWW')$, and since SW, SW' are perpendicular to each other, they are the bisectors of the angles at S[1]. According to a well-known theorem of plane geometry, we have for the triangles ABC and ABS,

$$\frac{CA}{CB}=\frac{AW}{BW}=\frac{AW'}{BW'};\ \frac{SA}{SB}=\frac{AW}{BW}=\frac{AW'}{BW'}.$$

Hence

$$\frac{SA}{SB}=\frac{CA}{CB}.$$

Now, if for any point S' in the plane we have $S'A/S'B = CA/CB$, the bisectors of the angles at S' will pass through W, W', according to the converse of the theorem cited, and the segment WW' will subtend at S' a right angle, *i. e.*, S' is on γ_3. These considerations may be repeated for the circles γ_1 and γ_2. Hence:

[1] John W. Russell, *Elementary Treatise on Pure Geometry*, pp. 18, 19.

The circle of Apollonius is the locus of a point the ratio of whose distances from two fixed points is constant.[1]

This property is frequently taken as the definition of the circle of Apollonius.

(3) From the definition and (1) it is evident that any two of the circles of Apollonius cross each other. Now, if I, I' are the points common to any two of these circles, say γ_2, γ_3, we have (2)

$$\frac{IA}{IC} = \frac{BA}{BC}; \quad \frac{IA}{IB} = \frac{CA}{CB}.$$

Hence

$$\frac{IB}{IC} = \frac{BA}{CA}, \text{ i. e., } I \text{ is on } \gamma_1.$$

The same being true for the point I', we infer that

The three circles of Apollonius have two points in common.[2]

COROLLARY. *The centers of the three circles of Apollonius are collinear.*

(4) The points A, B, W, W' being harmonic (2) and WW' being a diameter of γ_3 (1), any circle passing through A, B cuts the circle γ_3 orthogonally.[3] Similarly for γ_1 and γ_2. Hence:

The circumcircle is orthogonal to each of the three circles of Apollonius.

(5) The radii OA, OB, OC of the circumcircle ω are the tangents drawn from the center O of ω to the respective circles of Apollonius (1, 4), and since $OA = OB = OC$, the point O belongs to the radical axis of the circles $\gamma_1, \gamma_2, \gamma_3$, which is the line joining the points I, I' (3) common to the three circles. Consequently:

The common chord of the three circles of Apollonius passes through the center of the circumcircle.[2]

COROLLARY.[3] *The points of intersection M, M' of the circumcircle with the chord $s \equiv II'$ are harmonically separated by the points I, I'.*

(6) The circles ω and γ_3 having the point C in common (1) and being orthogonal (4), the tangent to ω at C passes through the center C_3 of γ_3. The centers C_1, C_2 of the circles γ_1, γ_2 are determined in a like manner. Hence:

The tangents to the circumcircle at the vertices of the triangle meet the opposite sides of the triangle in the centers of the respective circles of Apollonius.[2]

(7) The circles ω and γ_3 intersect in C (1); let C' denote their other point of intersection. Since OC, OC' are the tangents drawn from O to γ_3 (4), the chord $s_3 \equiv CC'$ is the polar of O with respect to γ_3, and therefore meets $s \equiv$ chord II' at the harmonic conjugate K of O with respect to the pair of points I, I'. Similarly for the chords of intersection $s_1 \equiv AA'$, $s_2 \equiv BB'$ of ω with the circles γ_1 and γ_2. Therefore:

The three chords joining the three pairs of points of intersection of the circum-

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[1] Weber and Wellstein, *Encyklopädie der Elementar-Matematik,* Vol. 2, p. 250, sec. ed.

[2] John Casey, *Analytic Geometry,* p. 146, sec. ed. This MONTHLY, February, 1915, p. 59.

[3] Russell, *loc. cit.,* p. 26.

circle with each of the three circles of Apollonius, meet on the radical axis of the Apollonian circles.[1]

The common point K is called the *Lemoine* or *Symmedian point* of the triangle; the chords s_1, s_2, s_3 are called the *Symmedian lines* or *Symmedians* of the triangle;[2] $s \equiv II'$ is called the *Brocard diameter.*[3]

(8) The tangents from C to ω are the lines C_3C and C_3C' (4), C_3 is therefore the pole of $s_3 \equiv CC'$ with respect to ω; since C_3 is on AB, the line s_3 passes through the pole of AB with respect to ω, which pole is the point of intersection of the tangents to ω at A and B. Similarly for the chords s_1, s_2. Hence:

The Symmedian lines pass through the respective vertices of the triangle formed by the tangents to the circumcircle at the vertices of the given triangle.[4]

(9) The four lines s_1, s_2, s_3, s meeting at K (7) have their respective poles C_1, C_2, C_3, ∞ (8) with regard to the circumcircle ω on the polar of K with respect to ω, and the anharmonic ratio of the four lines is equal to the anharmonic ratio of the four points.[5] Hence:

A. The line of centers of the Apollonian circles is the polar of the Lemoine point with respect to the circumcircle.

The line is called the *Lemoine axis* or *line.*[6]

B. The anharmonic ratio of the three Symmedians and the Brocard diameter is numerically equal to C_1C_3/C_2C_3.

(10) The Lemoine axis 1 is perpendicular to s at the mid-point J of the segment II' (3), and since the segments MM' and II' are harmonic (5, cor.), the point J lies without the segment MM'[7]. Consequently:

The points of intersection of the Lemoine axis with the circumcircle are always imaginary.

COROLLARY. *The Lemoine point of a triangle lies always within the circumcircle* (9*A*).

(11) Since the line s passes through the pole O, with regard to γ_3, of the chord CC' (7), the pole P_3 of s with respect to γ_3 lies on CC'. On the other hand P_3 lies on 1, since s is perpendicular to the diameter 1 of γ_3. Similarly for the poles P_1, P_2 of s with respect to γ_1, γ_2. Hence:

The Symmedians meet the Lemoine axis in the respective poles of the Brocard diameter with regard to the Apollonian circles.

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[1] William Gallatly, *The Modern Geometry of the Triangle*, p. 6.

[2] Casey, *loc. cit.*, pp. 63, 64; Gallatly, *loc. cit.*, pp. 1, 2; R. Lachlan, *An. Elem. Treatise on Pure Geometry*, pp. 62, 63.

[3] John Casey, *Analytical Geometry*, p. 146, sec. ed. This MONTHLY, Feb., 1915, p. 59.

[4] Gallatly, *loc. cit.*, p. 5.

[5] Russell, *loc. cit.*, pp. 165, 117.

[6] William Gallatly, *The Modern Geometry of the Triangle*, p. 6.

[7] Russell, *loc. cit.*, p. 15.

A TRIBUTE TO MILDRED LENORA SANDERSON.

The remarkable mathematical ability and originality shown by Miss Sanderson in her master's and doctor's theses and the very unusual ease with which she assimilated ideas in all branches of pure and applied mathematics, combined with her enthusiasm for that science, gave full promise of a highly successful career for her in research. Her death on October 15, 1914, only a year after completing her graduate studies, was not only a distinct loss to progress in mathematical research in America, but was a very keen blow to her fellow students, to all of whom she had endeared herself by her most lovable personality.

Miss Sanderson was born May 12, 1889, in Waltham, Mass., on the place where her ancestors had lived for over 200 years. She graduated from the North Grammar School at 13 and was valedictorian of her class in the Waltham High School. One of her instructors in the latter school writes that "Miss Sanderson was gentle-mannered, of brilliant intellect, an exact student, broad-minded, self-reliant, and courageous." She entered Mt. Holyoke in 1906, received "Sophomore Honors" in June, 1908, for general scholarship, and "Senior Honors" in mathematics at graduation in 1910. She held the Bardwell Memorial Fellowship for 1910 and 1911, and began her graduate work in mathematics at the University of Chicago, taking the degree A. M. in 1911. The subject of her master's thesis was "Generalizations in the Theory of Numbers and Theory of Linear Groups." Of this original and valuable thesis a very brief extract was printed in the *Annals of Mathematics*, Ser. 2, Vol. 13, 1911, pp. 36–39. This work might well have served for her doctor's thesis; but she was quite willing to undertake a new investigation in a wholly different field. The resulting thesis for the degree Ph.D., taken at the University of Chicago in 1913, was entitled "Formal Modular Invariants with Application to Binary Modular Covariants," and appeared in the *Transactions of the American Mathematical Society*, Vol. 14, 1913, pp. 489–500. This paper is a highly important contribution to this new field of work; its importance lies partly in the fact that it establishes a correspondence between modular and formal invariants. Her main theorem has already been frequently quoted on account of its fundamental character. Her proof is a remarkable piece of mathematics.

During the year 1911–13, she held a fellowship in mathematics in the University of Chicago. From October, 1913, to February, 1914, when her final illness began, she was instructor in mathematics in the University of Wisconsin.

If I may be permitted to add my personal tribute to the universally expressed tribute to her remarkable ability, it would be to say that she was my most gifted pupil.

L. E. DICKSON.

THE UNIVERSITY OF CHICAGO,
June, 1915.

BOOK REVIEWS.

Edited by W. H. Bussey, University of Minnesota.

Plane Geometry. By John H. Williams and Kenneth P. Williams. Lyons and Carnahan, Chicago, 1915. 264 pages.

At the outset an effort is made to justify the study of geometry. This is usually a vital question in the minds of young people when they undertake the subject for the first time.

Scattered through the book will be found exercises in construction, numerical problems, and original exercises. These appear as soon as theorems or definitions have been given that lead up to them. Numerical exercises and original theorems, if brought in as soon as possible, will aid materially in fixing fundamental notions without, at the same time, taking up the usual theorems too rapidly. In this respect the book is well conceived. Following the definitions of complementary and supplementary angles on page 21, is found a well-chosen list of numerical problems, also following the theorem on the "sum of the angles of a triangle" will be found appropriate numerical exercises.

At the close of each book is a review which summarizes, in concise form, the main notions of the book. This includes definitions, axioms, postulates, corollaries, theorems, etc., in question form. For example: When is one proposition the converse of another? How are triangles classified as to their sides? How are the acute angles of a right triangle related? These and many others constitute a complete summary of the first book. This book closes with the significant statement: "Name three general reasons for studying geometry."

It would seem that the field for plane geometry had been so thoroughly covered that there could be no excuse for new texts; however modern conditions and modern thought call for some changes. The school curriculum is more crowded than formerly and pupils take up the study of geometry at a somewhat earlier age; and the tendency toward the practical makes it necessary for a geometry text to justify itself from the standpoint of practicability, besides being somewhat simplified to meet these changed conditions. To hold its own with the pressing demands of other subjects, geometry must be made interesting. Variables and limits are not introduced in Book II, as is usual, but instead in Book V just before the theorems on circumference and area of circles. This arrangement seems very satisfactory; the theory of limits is needed only in connection with circles. Earlier theorems on incommensurables may with propriety be omitted.

On page 88 is found the following corollary: "The ratio of two incommensurable arcs of the same circle or of equal circles can be expressed as the ratio of two commensurable arcs to any degree of exactness, if a sufficiently small unit of measure is used;" with a note to the effect that in practice there is no distinction to be made between commensurable and incommensurable lines as the fraction to be dropped from one of two incommensurable lines to make the lines commensurable is less than the error made in the work of measuring.

On the whole this text seems well adapted to present needs and should meet with a generous response.

NORTH HIGH SCHOOL,
MINNEAPOLIS.

F. W. GATES.

New Plane Geometry. By EDWARD RUTLEDGE ROBBINS. American Book Company, New York, 1915. 264 pages.

This book is a revision of a Plane Geometry published by the same author a few years ago. The revision, as the author suggests, is an outgrowth of the author's experience and suggestions from teachers who have used the former edition as well as from recommendations of the "National Committee of Fifteen."

While the general plan of the new book is much the same as the old, there are many additions that make the book comply with recent demands.

Each page is attractive, the material is well arranged, statements of theorems and the words *given, to prove, proof,* etc., are in italics.

Simple fundamental truths are explained instead of being formally demonstrated," as for example, "all straight angles are equal." Several such will be found on pages 13 and 14.

No theorems are demonstrated in full. Proofs are given in outline, reasons indicated by references. It would seem that if a few theorems were demonstrated in full the pupil might have a more definite notion of what constitutes a complete demonstration.

Original exercises are scattered through the book in abundance. These appear as early as possible. After theorem (2) will be found eleven original exercises; after theorem (34), 28 exercises; after theorem (35), 11 more; and so on.

These exercises do not depend upon previous exercises for proofs, the numbered references alone being sufficient.

The large number of original exercises makes it possible to furnish the ambitious pupil with an abundance of choice material.

In all there are about 200 original exercises in Book I, especially of the theorem variety. Very few numerical problems are to be found in this book; possibly a number of such problems would have been an advantage.

Book II is unusually attractive. Definitions are well illustrated. A large number of original exercises are found in this book. These must be seen to be appreciated.

At the close of Book III will be found a collection of 69 exercises (numerical). These involve the application of a large number of theorems of every description. These exercises are followed by 53 original theorems, and these by 43 original constructions.

At intervals are historical notes which help to interest the pupil in the subject.

Summaries found on pages 68, 94, and 180 classify the theorems in a way that will be helpful for reference purposes.

This text seems to be an improvement over the author's earlier edition, both in attractiveness of the printed page and abundance of well-selected original exercises. After all the pupil's ability to solve original theorems is the real test of geometrical knowledge.

The text as a whole seems to meet very well the recent demands for the subject.

NORTH HIGH SCHOOL,
MINNEAPOLIS.

F. W. GATES.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

PROBLEMS FOR SOLUTION.

ALGEBRA.

441. Proposed by W. D. CAIRNS, Oberlin College.

Prove that the equation $(e - 1)x = e^x - 1$ has two and only two real roots. [Adapted from *L'Intermédiaire*.]

442. Proposed by CLIFFORD N. MILLS, Brookings, North Dakota.

Show that the sum of n terms of the series $1/2 - 1/3 + 1/4 - 1/6 + 1/8 - 1/12 + \cdots$ is $1/3[1 - (1/2)^{n/2}]$ when n is even, and $1/3[1 + 2\sqrt{2}(1/2)^{(n/2)+1}]$ when n is odd.

GEOMETRY.

472. Proposed by PAUL CAPRON, U. S. Naval Academy.

The sides of a spherical triangle are a, b, c; the corresponding opposite angles are A, B, C; p and P are the polar distances of the inscribed and circumscribed circles; $a + b + c = 2s$; $A + B + C = 2S$. From a geometric figure, by the formula for solving right spherical triangles, show that

$$\tan^2 p = \operatorname{cosec} s \sin (s - a) \sin (s - b) \sin (s - c); \tag{1}$$

$$\cot^2 P = - \sec S \cos (S - A) \cos (S - B) \cos (S - C). \tag{2}$$

Thus establish the usual formulas for the tangent of the half-sides and half-angles.

Also show that

$$\frac{\text{sine of angle}}{\text{sine of the opposite side}} = \frac{\cot P \cos S}{\tan p \sin s}. \tag{3}$$

473. Proposed by FRANK R. MORRIS, Gendale, Calif.

What is the length of the longest rectangle an inch wide that can be placed inside another rectangle 12 inches long and 8 inches wide. Obtain the result correct to the third decimal.

CALCULUS.

393. Proposed by LAENAS G. WELD, Pullman, Ill.

Find the area of the least ellipse which can be drawn upon the face of a brick wall so as to inclose four bricks.

394. Proposed by W. W. BURTON, Macon, Ga.

A horse runs 10 miles per hour on a circular race-track in the center of which is an arc-light. How fast will his shadow move along a straight board fence (tangent to the track at the starting point) when he has completed one eighth of the circuit?

MECHANICS

[Numbers 307 and 308 were omitted in the May issue.]

307. Proposed by LAENAS G. WELD, Pullman, Illinois.

Four forces W, X, Y, and Z, concurrent in O, are in equilibrium. Prove that

$$W : X : Y : Z :: \Delta_1 : \Delta_2 : \Delta_3 : \Delta_4,$$

where

$$\Delta_1 = \begin{vmatrix} 1 & \cos XOY & \cos XOZ \\ \cos XOY & 1 & \cos YOZ \\ \cos XOZ & \cos YOZ & 1 \end{vmatrix}^{\frac{1}{2}}; \quad \Delta_2 = \begin{vmatrix} 1 & \cos WOY & \cos WOZ \\ \cos WOY & 1 & \cos YOZ \\ \cos WOZ & \cos YOZ & 1 \end{vmatrix}^{\frac{1}{2}};$$

$$\Delta_3 = \begin{vmatrix} 1 & \cos WOX & \cos WOZ \\ \cos WOX & 1 & \cos XOZ \\ \cos WOZ & \cos XOZ & 1 \end{vmatrix}^{\frac{1}{2}}; \quad \Delta_4 = \begin{vmatrix} 1 & \cos WOX & \cos WOY \\ \cos WOX & 1 & \cos XOY \\ \cos WOY & \cos XOY & 1 \end{vmatrix}^{\frac{1}{2}}.$$

308. Proposed by H. S. UHLER, Yale University.

Prove that when a ray of light passes obliquely through a prism in such a manner as to maintain a constant value for the total deviation of the projection of the ray on a principal section, the ray inside the prism generates a cone of elliptical right section. It is assumed that the prism is surrounded by a medium having a smaller index of refraction than the index of the material of the prism.

NUMBER THEORY.

234. Proposed by FRANK IRWIN, University of California.

Start with any number, for instance 89, and add to it the number obtained by reversing the order of its digits: $89 + 98 = 187$. Now perform the same operation on the result: $187 + 781 = 968$. If we continue in this way we arrive, after a certain number of operations, at a number which reads the same forwards and backwards (24 operations bring us to 8813200023188). Will this be the case no matter with what number we start?

Note.—I am told that this is an old problem, but do not know whether it has ever been solved. (No other number under 100, except, of course, 98, requires so many operations to lead to the desired result, as 89 does.)

SOLUTIONS OF PROBLEMS.

ALGEBRA.

409. Proposed by C. E. GITHINS, Wheeling, W. Va.

Find integral values for the edges of a rectangular parallelepiped so that its diagonal shall be rational.

(*A*) REMARKS BY ARTEMAS MARTIN, Washington, D. C.

I. On page 162 of the May MONTHLY, Mr. Eells says: "I fail to see, however, how I solved a *different* problem from the one porposed."

The problem proposed is stated above; the problem solved by Mr. Eells follows:

Find integral values for the edges of a rectangular parallelepiped so that its diagonal, and the diagonal of one of its faces, shall be rational.

The problem proposed does not require the diagonal of one of the faces of the solid to be rational, and the added condition restricts its generality and limits the number of possible solutions.

II. Mr. Eells says on page 269 of the October issue, without any qualifying condition: "This gives the smallest rational parallelepiped," edges 3, 4, 12 and diagonal 13.

III. The fault in my solution was in assuming $x = a,\ y = b,\ d = z + c$. I should have put $x = na,\ y = nb$ and $d = z + nc$; then

$$(na)^2 + (nb)^2 + z^2 = (z + nc)^2 = z^2 + 2ncz + n^2c^2,$$

which gives, after dividing by n,

$$z = \frac{na^2 + nb^2 - nc^2}{2c} \quad \text{and} \quad d = \frac{na^2 + nb^2 + nc^2}{2c}.$$

Now take $n = 2c$ and we have the integral values

$$x = 2ac,\ y = 2bc,\ z = a^2 + b^2 - c^2,\ d = a^2 + b^2 + c^2;$$

therefore

$$(a^2 + b^2 + c^2)^2 = (a^2 + b^2 - c^2)^2 + (2ac)^2 + (2bc)^2 \qquad (A),$$

whatever be the values of a, b, c; and it is seen, without any elaborate proof, that x and y are always even numbers.

Permuting the letters a, b, c in (A) we get

$$\begin{aligned}(a^2 + b^2 + c^2)^2 &= (a^2 + b^2 - c^2)^2 + (2ac)^2 + (2bc)^2 \\ &= (a^2 + c^2 - b^2)^2 + (2bc)^2 + (2ab)^2 \\ &= (b^2 + c^2 - a^2)^2 + (2ab)^2 + (2ac)^2.\end{aligned}$$

The additional formulas, however, will only give sets of values for x, y, z that can be obtained by interchanging the numerical values assigned to a, b, c in (A).

In order to obtain values for x, y, z that have no common divisor it is necessary to impose certain restriction on a, b, c. Such values of x, y, z may be called a prime set.

1. The values given to a, b, c must not *contain* a common factor. Hence they can not all be even as in that case x, y, z would be divisible by 2.
2. If a, b, c be all odd, the numbers x, y, z will not have a common divisor.
3. If any two are even and the other odd, the numbers will not have a common divisor.
4. If any two are odd and the other even, then 2 will be even and the numbers will have the common divisor 2 since x and y are always even.
5. But the numbers will be a prime set in all cases after the common factor is divided out.

IV. Mr. Carleton does not put down the rational value of c in his solution on page 164, but it is obvious that

$$c = \sqrt{(2ab)} = \sqrt{(2a \times \tfrac{1}{2}a^3)} = a^2,$$

and that we have

$$(a)^2 + (a^2)^2 + (\tfrac{1}{2}a^3)^2 = a^2 + a^4 + \tfrac{1}{4}a^6 = a(1 + \tfrac{1}{2}a^2)^2,$$

and the numbers (a, b, c) have a common factor a; therefore

$$1^2 + a^2 + \tfrac{1}{4}a^4 = (1 + \tfrac{1}{2}a^2)^2,$$

whatever be the value of a; but, for integers, a must be even.

Fractional values could be avoided by putting $a = 2m^2,\ b = n^2$; for we would then have

$$c = \sqrt{(2ab)} = \sqrt{(4m^2n^2)} = 2mn,$$

and

$$(2m^2)^2 + (2mn)^2 + (n^2)^2 = (2m^2 + n^2)^2,$$

where m and n must be prime to each other.

See the *Mathematical Magazine*, Vol. II., No. 5 (Washington, D. C., Oct., 1891), page 72, where the writer published this method of solution. For other solutions, see the same No., pp. 71–75.

V. It has been claimed that the equation (A) above includes all possible solutions of

$$x^2 + y^2 + z^2 = 0,$$

but the writer is not at present prepared to affirm or deny that claim.

(B) REMARKS BY J. W. YOUNG, Dartmouth College.

After reading the remarks by W. C. Eells, on Algebra Problem 409, in the April number of the MONTHLY, it occurred to me that the geometric formulation for the right-triangle problem given by Klein in his *Elementarmath. v. höheren Standp. aus*, Vol. I, should yield a general solution also for the rectangular parallelepiped case. The problem is to solve in integers the equation

$$a^2 + b^2 + c^2 = d^2 \quad (> 0). \tag{1}$$

Placing $x = \frac{a}{d}$, $y = \frac{b}{d}$, $z = \frac{c}{d}$ the equation becomes

$$x^2 + y^2 + z^2 = 1. \tag{2}$$

Equation (1) will be completely solved, if we find all *rational* points on the sphere (2). $(-1, 0, 0)$ is one such point. Any straight line through $(-1, 0, 0)$ and any other rational point will have the equations

$$\frac{x+1}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu}, \tag{3}$$

with λ, μ, ν integral and without a common factor; and conversely, every such line (3) will cut out a rational point, P, besides $(-1, 0, 0)$. Solving (2) and (3) we find

$$P = \left(\frac{\lambda^2 - \mu^2 - \nu^2}{\lambda^2 + \mu^2 + \nu^2}, \frac{2\mu\lambda}{\lambda^2 + \mu^2 + \nu^2}, \frac{2\nu\lambda}{\lambda^2 + \mu^2 + \nu^2}\right).$$

If D represents the H. C. F. of $\lambda^2 - \mu^2 - \nu^2$, $2\mu\lambda$, $2\nu\lambda$, and $\lambda^2 + \mu^2 + \nu^2$, then the *general solution* of (1) is given by

$$(a, b, c, d) = \left(\frac{k}{D}(\lambda^2 - \mu^2 - \nu^2), \frac{k}{D}\cdot 2\mu\lambda, \frac{k}{D}\cdot 2\nu\lambda, \frac{k}{D}(\lambda^2 + \mu^2 + \nu^2)\right), \tag{4}$$

where k is any integer.

Y. A. Le Besgue's solutions, to which reference has already been made by Mr. Eells, yield, for $\delta = 0$, the solutions above for $K = D$. If λ, μ, ν are, *e. g.*, relatively prime, no solutions are duplicated by (4). Moreover it is then easy to see that D is equal to the H. C. F. of λ and $\mu^2 + \nu^2$ or twice this H. C. F. The above method will clearly hold for the general equation $\sum_{i=1}^{n} x_1{}^2 = n^2$.

420. Proposed by ELBERT H. CLARKE, Purdue University.

Given the infinite series,

$$\frac{a}{r} + \frac{b}{r^2} + \frac{a+b}{r^3} + \frac{a+2b}{r^4} + \frac{2a+3b}{r^5} + \cdots,$$

in which a and b are any numbers, and where each numerator after the second is the sum of the two preceding numerators. To find the region of convergence and the sum of the series.

This problem is a generalization of one solved in the January (1914) number of the MONTHLY.

II. SOLUTION BY THE PROPOSER.

Let us designate the above numerators as follows:

$$U_1 = a;\ U_2 = b;\ U_3 = a + b;\ U_n = U_{n-1} + U_{n-2}.$$

$$\sum_{n=1}^{\infty} \frac{U_n}{r^n} = \frac{1}{\sqrt{5}} \left(\frac{a + b\rho_1}{\rho_1(r - \rho_1)} - \frac{1}{\sqrt{5}} \frac{a + b\rho_2}{\rho_2(r - \rho_2)} \right).$$

The right-hand member now reduces to

$$\frac{b + ar - a}{r^2 - r - 1}$$

when ρ_1 is replaced by $(1 + \sqrt{5})/2$ and ρ_2 by $(1 - \sqrt{5})/2$.

Note.—We are publishing this solution for the reason that the previously published solution referred to did not consider the question of convergency correctly, and the proper investigation of this question was the Proposer's chief reason for proposing the problem. EDITORS.

ALGEBRA.

429. Proposed by C. N. SCHMALL, New York City.

It is given that d_1, d_2, d_3 are the greatest common divisors of y and z, z and x, x and y, respectively; also that m_1, m_2, m_3 are the least common multiples of the same pairs of members. If d and m are the greatest common divisor and least common multiple, respectively, of x, y, and z, show that

$$\frac{m}{d} = \left(\frac{m_1 m_2 m_3}{d_1 d_2 d_3} \right)^{\frac{1}{2}}.$$

SOLUTION BY FRANK IRWIN, University of California.

It is evident that we can get the least common multiple of two numbers by dividing their product by their greatest common divisor:

$$m_1 = \frac{yz}{d_1}, \quad m_2 = \frac{xz}{d_2}, \quad m_3 = \frac{xy}{d_3}.$$

Similarly with the three numbers x, y, z, if we divide their product by $d_1 d_2 d_3$, we should have their least common multiple, except that we have divided out d once too often:

$$m = \frac{xyz}{d_1 d_2 d_3} \cdot d.$$

We have then:

$$\left(\frac{m_1}{d_1} \cdot \frac{m_2}{d_2} \cdot \frac{m_3}{d_3} \right)^{\frac{1}{2}} = \left(\frac{yz}{d_1^2} \cdot \frac{zx}{d_2^2} \cdot \frac{xy}{d_3^2} \right)^{\frac{1}{2}} = \frac{xyz}{d_1 d_2 d_3} = \frac{m}{d}.$$

Also solved by A. H. HOLMES, ELMER SCHUYLER, G. W. HARTWELL, FRANK R. MORRIS, N. P. PANDYA, HERBERT N. CARLETON, PAUL CAPRON, J. A. CAPARO, and the PROPOSER.

GEOMETRY.

443. Proposed by C. N. SCHMALL, New York City.

A quadrilateral of any shape whatever is divided by a transversal into two quadrilaterals. The diagonals of the original figure and those of the two resulting (smaller) figures are then drawn. Show that their three points of intersection are collinear.

III. SOLUTION BY LAENAS G. WELD, Pullman, Ills.

To the triangle ABC draw the transversals MN, intersecting AB in M and AC in N, and QR intersecting AB in Q and AC in R. Then $BCRQ$, $BCNM$ and

$QRNM$ are the three quadrilaterals in question. Designate the intersection of the diagonals BN and CM by F, that of BR and CQ by G, and that of QN and RM by H. Then F, G, and H are collinear?

In vectors, let $AB = \beta$, $AC = \gamma$; $AM = m\beta$, $AN = n\gamma$; $AQ = q\beta$, $AR = r\gamma$. Then

$$MC = -m\beta + \gamma \quad \text{and} \quad MF = x(-m\beta + \gamma),$$

in which x is to be determined. Also

$$NB = -n\gamma + \beta \quad \text{and} \quad NF = y(-n\gamma + \beta),$$

in which y is to be determined. Now

$$m\beta + x(-\beta m + \gamma) = AF = n\gamma + y(-n\gamma + \beta); \tag{1}$$

or

$$m(1-x)\beta + x\gamma = n(1-y)\gamma + y\beta,$$

in which the coefficients of β and γ may be equated. Thus

$$mx + y = m,$$

$$x + ny = n.$$

Solving:

$$x = \frac{n(m-1)}{mn-1}, \quad y = \frac{m(n-1)}{mn-1};$$

whence, by Eq. (1),

$$AF = \frac{m(n-1)}{mn-1}\cdot\beta + \frac{n(m-1)}{mn-1}\cdot\gamma. \tag{2}$$

Similarly:

$$AG = \frac{q(r-1)}{qr-1}\cdot\beta + \frac{r(q-1)}{qr-1}\cdot\gamma, \tag{3}$$

and

$$AH = \frac{mq(n-r)}{mn-qr}\cdot\beta + \frac{nr(m-q)}{mn-qr}\cdot\gamma. \tag{4}$$

By the theory of vectors, if there are three multipliers, λ, μ, ν, such that

$$\lambda\cdot AF + \mu\cdot AG + \nu\cdot AH = 0,$$

while

$$\lambda + \mu + \nu = 0,$$

the vectors AF, AG, AH terminate collinearly. This requires, by Equations (2), (3), (4), the consistency of the equations

$$\frac{m(n-1)}{mn-1}\cdot\lambda\beta + \frac{q(r-1)}{qr-1}\cdot\mu\beta + \frac{mq(n-r)}{mn-qr}\cdot\nu\beta = 0,$$

$$\frac{n(m-1)}{mn-1}\cdot\lambda\gamma + \frac{r(q-1)}{qr-1}\cdot\mu\gamma + \frac{nr(m-q)}{mn-qr}\cdot\nu\gamma = 0,$$

$$\lambda + \mu + \nu = 0.$$

This consistency is established by the vanishing of the determinant

$$\Delta \equiv \begin{vmatrix} m(n-1) & q(r-1) & mq(n-r) \\ n(m-1) & r(q-1) & nr(m-q) \\ mn-1 & qr-1 & mn-qr \end{vmatrix};$$

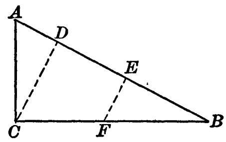
which proves the proposition.

Note.—Solutions of this problem appeared in the January and May numbers of the MONTHLY, presenting two different methods of attack. The one here given presents still a different method.

EDITORS.

459. Proposed by C. N. SCHMALL, New York City.

In a right triangle ABC, right-angled at C, a point F is taken in the side CB, and perpendiculars CD and FE are dropped on the hypothenuse AB.



Prove that

$$AD \cdot AE + CD \cdot EF = \overline{AC}^2.$$

SOLUTION BY HERBERT N. CARLETON, West Newbury, Mass.

Triangles ACD, CDB, EFB, and ACB are similar. Hence,

$$AD : CD = EF : EB;$$

whence

$$AD \cdot EB = CD \cdot EF.$$

Similarly

$$AD : AC = AC : AB;$$

whence

$$\overline{AC}^2 = AB \cdot AD = (AE + EB)AD = AD \cdot AE + AD \cdot EB.$$

Substituting for $AD \cdot EB$, we have

$$\overline{AC}^2 = AD \cdot AE + CD \cdot EF.$$

Also solved by A. M. HARDING, J. W. CLAWSON, ELMER SCHUYLER, FRANK IRWIN, WALTER C. EELLS, L. G. WELD, J. A. CAPARO, A. H. HOLMES, NATHAN ALTSHILLER, G. W. HARTWELL, HORACE OLSON, and HARMON ANNING.

460. Proposed by J. W. CLAWSON, Ursinus College, Pa.

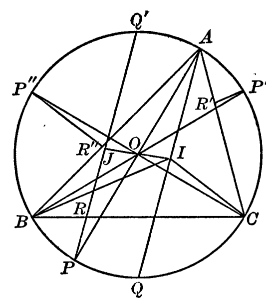
ABC is a triangle, O and I the centers of the circum- and in-circles respectively, and I', I'', I''' the centers of the three escribed circles. If AO, BO, CO meet the circumcircle in P, P', P'' respectively, and PR, $P'R'$, $P''R''$ are drawn parallel respectively to AI, BI, CI to meet BC, CA, AB respectively in R, R', R'', prove that: (1) PR, $P'R'$, $P''R''$ are concurrent, say at J. (2) $JO = OI$. (3) JI', JI'', JI''' are perpendicular respectively to BC, CA, AB. (4) AR, BR', CR'' are concurrent.

I. SOLUTION BY THE PROPOSER.

(1) Extend IO its own length to J. Join JP. Then $\triangle$'s AOI, POJ are congruent. Hence, $\angle APJ = \angle PAI$. Hence J lies on PR. Similarly, J can be shown to lie on $P'R'$ and on $P''R''$.

(2) By construction, $JO = OI$.

(3) Extend AI to meet the circumcircle at Q. Then Q is the center of a circle passing through B, C, I, I'. Hence, Q bisects II'. Also O bisects IJ. Hence, $I'J$ is parallel to OQ. But OQ is perpendicular to BC, since arc BQ = arc QC. Therefore, JI' is perpendicular to BC.



Similarly, JI'' and JI''' are perpendicular to CA and AB, respectively.

(4) Extend PJ to meet the circumcircle at Q'. It is easily shown that Q' is the opposite extremity of the diameter through Q. Then $\angle BPQ' = \angle Q'PC$.

$$\therefore \quad \frac{BR}{RC} = \sqrt{\frac{4r^2 - c^2}{4r^2 - b^2}}, \qquad \frac{CR'}{R'A} = \sqrt{\frac{4r^2 - a^2}{4r^2 - c^2}}, \qquad \frac{AR''}{R''B} = \sqrt{\frac{4r^2 - b^2}{4r^2 - a^2}},$$

where r = radius of circumcircle, and a, b, c are the sides of the triangle.

Multiplying,

$$\frac{BR \cdot CR' \cdot AR''}{RC \cdot R'A \cdot R''B} = 1.$$

Hence, by the converse of Ceva's Theorem, AR, BR,' CR'' are concurrent.

II. Solution by N. P. Pandya, Sojitra, India.

(1) The quadrilateral $ABPP'$, having as diagonals diameters of the circumcircle, is a rectangle. Hence $AB = PP'$, $AB \parallel PP'$; and similarly,

$$BC = P'P'', \; BC \parallel P'P'', \; CA = P'P'' \text{ and } CA \parallel PP''.$$

$\therefore$ $\triangle$s $PP'P''$ and ABC are congruent and have their sides respectively parallel.

$\therefore$ $\angle P'PP'' = \angle ABC$ and PR bisects $\angle P'PP''$.

Similarly, $P'R'$ and $P''R''$ are bisectors of the remaining angles of $\triangle PP'P''$. Hence, PR, $P'R'$, $P''R''$ are concurrent at J.

(2) As $AB, PP'; BC, P'P''; CA, P''P$ are symmetrically situated with respect to O, then I and J are also so situated; hence, $IO = JO$, J being the incenter of $\triangle PP'P''$.

(3) Let the circumcircle cut II' at L. Then $\angle I'IC = \angle IAC + \angle ACI = \angle BAL + \angle ICB = \angle BCL + \angle ICB = \angle ICL$. Hence, since $\angle ICI'$ is a right angle, L is the midpoint of II'; and O is the midpoint of IJ. Hence, JI' is parallel to OL. But as $\angle BAL = \angle LAC$, arc BL = arc LC. Hence,

$$OL \perp BC \text{ and } \therefore JI' \perp BC.$$

Similarly, $JI'' \perp CA$ and $JI''' \perp AB$.

(4) Arc $P''B$ = arc $P'C$. Hence, $\angle P''PB = \angle P'PC$; hence $\angle BPR = \angle RPC$. Hence, $BR : RC = BP : PC$. Similarly, $CR' : R'A = CP' : P'A$, and $AR'' : R''B = AP'' : P''B$. Hence,

$$\frac{BR \cdot CR' \cdot AR''}{RC \cdot R'A \cdot R''B} = \frac{BP \cdot CP' \cdot AP''}{PC \cdot P'A \cdot P''B}.$$

But $BP = AP'$, being opposite sides of a rectangle. Similarly, $CP' = BP''$ and $AP'' = CP$. Hence,

$$\frac{BR \cdot CR' \cdot AR''}{RC \cdot R'A \cdot R''B} = 1,$$

numerically. Hence, AR, BR', CR'' are concurrent.

Also solved by A. M. Harding.

CALCULUS.

373. Proposed by C. N. SCHMALL, New York City.

In the *Encyclopaedia Britannica* article on "Capillary Action" (Vol. 5, p. 268, 11th ed.) it is shown that $1/R_1 + 1/R_2 = p/T$, in the case of a soap bubble, where R_1, R_2 are the *principal radii of curvature* at any point of the bubble; p, the difference of air-pressure; T, the energy per unit area of the film. Employing the principle that the soap bubble tends to assume a form such that the area of its surface is a *minimum* for a *given volume* of air, show by the calculus of variations that $1/R_1 + 1/R_2 = k$, a constant.

Solution by the Proposer.

We have here to determine the solid which, with a given volume (capacity), contains the least surface. Hence, we have to make the surface integral

$$U = \iint \sqrt{1 + p^2 + q^2}\, dxdy \tag{1}$$

a minimum, subject to the condition that the volume integral

$$I = \iint z\, dxdy \tag{2}$$

is constant.

It should be remembered that in this discussion $p = \partial z/\partial x$, $q = \partial z/\partial y$, $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x \partial y$, and $t = \partial^2 z/\partial y^2$.

Let k be a constant. Then it is evident that, when the surface (1) is a minimum, the binomial

$$\iint \sqrt{1+p^2+q^2}\, dxdy + \iint kz\, dxdy \equiv \iint (\sqrt{1+p^2+q^2} + kz) dxdy \equiv \iint V\, dxdy \tag{3}$$

will also be a minimum.

Now, taking x and y as the independent variables, the condition for a mini-

mum is, by the calculus of variations,

$$\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = L, \tag{4}$$

(TODHUNTER'S *Int. Cal.*, p. 374, 5th ed.)

where

$$M = \frac{\partial V}{\partial p} = \frac{p}{\sqrt{1 + p^2 + q^2}},$$

$$N = \frac{\partial V}{\partial q} = \frac{q}{\sqrt{1 + p^2 + q^2}},$$

$$L = \frac{\partial V}{\partial z} = k.$$

(TODHUNTER'S, *Int. Cal.*, p. 372.)

Hence, equation (4) becomes

$$\frac{\partial}{\partial x}\left(\frac{p}{\sqrt{1 + p^2 + q^2}}\right) + \frac{\partial}{\partial y}\left(\frac{q}{\sqrt{1 + p^2 + q^2}}\right) = k \tag{5}$$

or,

$$\frac{r(1 + p^2 + q^2) - (pr + qs)p + t(1 + p^2 + q^2) - (ps + qt)q}{(1 + p^2 + q^2)^{\frac{3}{2}}} = k,$$

or,

$$\frac{(1 + q^2)r - 2pqs + (1 + p^2)t}{(1 + p^2 + q^2)^{\frac{3}{2}}} = k, \tag{6}$$

which is the partial differential equation of the required minimal surface, the integral of which will represent the surface itself.

Again, R_1 and R_2 are known as the *principal radii of curvature* at any point of the surface. The equation giving these is

$$(rt - s^2)\,R^2 - \sqrt{1 + p^2 + q^2}\,[(1 + p^2)t - 2pqs + (1 + q^2)r]R + (1 + p^2 + q^2)^2 = 0. \tag{7}$$

(GOURSAT-HEDRICK'S *Math. Anal.*, Vol. 1, p. 504, Eq. 13.)

If R_1, R_2, be the roots of this quadratic in R, we have

$$R_1 + R_2 = \frac{\sqrt{1 + p^2 + q^2}[(1 + p^2)t - 2pqs + (1 + q^2)r]}{rt - s^2}, \tag{8}$$

$$R_1R_2 = \frac{(1 + p^2 + q^2)^2}{rt - s^2}. \tag{9}$$

Dividing (8) by (9), we obtain

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{(1 + p^2)t - 2pqs + (1 + q^2)r}{(1 + p^2 + q^2)^{\frac{3}{2}}}. \tag{10}$$

(See EISENHART'S *Diff. Geom.*, p. 126, ex. 3.)

Comparing equations (6) and (10) we have the required result,

$$\frac{1}{R_1} + \frac{1}{R_2} = k.$$

Note 1.—The sphere,

$$x^2 + y^2 + z^2 = a^2 \tag{11}$$

and the cylinder,

$$y^2 + z^2 = b^2, \tag{12}$$

are examples of minimal surfaces satisfying Eq. (6).

Thus, in (11),

$$p = -x/z, \qquad q = -y/z, \qquad \sqrt{1+p^2+q^2} = a/z,$$

$$\therefore \quad M = -x/a, \qquad N = -y/a,$$

and (11) becomes $k + 2/a = 0$; and (12), $k + 1/b = 0$.

Note 2.—In the foregoing solution I have utilized the notation employed in the chapter on the Calculus of Variations in TODHUNTER's *Integral Calculus*, fifth edition.

374. Proposed by C. N. SCHMALL, New York City.

Show that, on a *Mercator's Chart*, a great circle of a sphere of radius r_1 will be represented by a curve whose equation is of the form

$$c(e^{y/r} - e^{-(y/r)}) = 2 \sin\left(\frac{x}{r} + \theta\right).$$

I. SOLUTION BY ELIJAH SWIFT, University of Vermont.

If the latitude and longitude on the above sphere be denoted by the letters φ and θ respectively, θ varying from 0° to 360°, and φ from $-90°$ to $+90°$; then if axes be taken with origin at the center of the sphere with the xy-plane as the plane of the equator, and if longitude be measured from the x-axis, we have for any point on the sphere

$$x = r\cos\varphi\cos\theta, \; y = r\cos\varphi\sin\theta, \; z = r\sin\varphi.$$

The equation of a great circle is obtained by substituting these values in the equation of any diametral plane, $Ax + By + Cz = 0$, and is

$$A\cos\varphi\cos\theta + B\cos\varphi\sin\theta + C\sin\varphi = 0. \tag{1}$$

The sphere is mapped on a *Mercator's Chart* by taking a cylinder tangent to the sphere along the equator and projecting a meridian ($\theta =$ const.) on a generating line of the cylinder.

Any point on the sphere on this meridian has for its image on the chart a point on the corresponding generating line at a distance $r \log\tan(\pi/4 + \varphi/2)$ from the equator.

When we develop the cylinder on a plane, we can choose axes in that plane so that the coördinates of this point are

$$x = r\theta, \qquad y = r\log\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right).$$

Solving these equations for θ and φ, $\theta = x/r$, $\varphi = 2\arctan(e^{y/r}) - \pi/2$. Substituting these values in (1), we obtain

$$A\cos\frac{x}{r} + B\sin\frac{x}{r} + C\tan\left\{2\arctan e^{y/r} - \frac{\pi}{2}\right\} = 0,$$

which reduces at once to the form given, if we let $c = -C/\sqrt{A^2+B^2}$, and $\sin\theta = A/\sqrt{A^2+B^2}$.

II. SOLUTION BY A. M. HARDING, University of Arkansas.

The equation of the sphere is

$$\frac{x}{r} = \cos u \cos v, \qquad \frac{y}{r} = \cos u \sin v, \qquad \frac{z}{r} = \sin u.$$

Any plane through the center will be given by $ax + by + cz = 0$, or

$$a \cos u \cos v + b \cos u \sin v + c \sin u = 0.$$

Dividing by $\sqrt{a^2 + b^2}$, we obtain $\sin(v + \theta) + (c/\sqrt{a^2 + b^2}) \tan u = 0$, where θ is defined by the equations; $\sin \theta = a/\sqrt{a^2 + b^2}$, $\cos \theta = b/\sqrt{a^2 + b^2}$.

The transformation is

$$\frac{y}{r} = \operatorname{logtan}\left(\frac{u}{2} + \frac{\pi}{4}\right), \qquad \frac{x}{r} = v.$$

Solving the first of these equations for $\tan u$, we obtain

$$\tan u = \frac{e^{y/r} - e^{-y/r}}{2}.$$

Hence, by substitution, we have

$$-\frac{c}{\sqrt{a^2 + b^2}}(e^{y/r} - e^{-y/r}) = 2 \sin\left(\frac{x}{r} + \theta\right),$$

which is of the required form.

Also solved by PAUL CAPRON and the PROPOSER.

MECHANICS.

278. Proposed by A. M. HARDING, University of Arkansas.

A spherical shell of mass m explodes when moving with negligible velocity at a height of h feet above the ground. The shell is divided into very small particles, each of which moves, after the explosion, away from the center of the shell with a speed v, and ultimately falls to the ground. Find the total mass of the fragments which will be found per unit area at any specified distance from the point vertically underneath the shell.

SOLUTION BY H. S. UHLER, Yale University.

Let θ and φ denote the angles which the two trajectories, passing through the same point on the ground, make at the instant of the explosion, with the negative and positive directions of the axis of h respectively. φ must be acute but θ may be obtuse. The familiar equation $x = vt + \frac{1}{2}at^2$ leads to

$$r = \frac{v \sin \theta}{g}(+\sqrt{v^2 \cos^2 \theta + 2gh} - v \cos \theta), \tag{1}$$

$$r = \frac{v \sin \varphi}{g}(+\sqrt{v^2 \cos^2 \varphi + 2gh} + v \cos \varphi). \tag{2}$$

Rationalization of (1) and (2) gives

$$gr^2 + 2rv^2 \sin \theta \cos \theta - 2hv^2 \sin^2 \theta = 0, \tag{3}$$

$$gr^2 - 2rv^2 \sin \varphi \cos \varphi - 2hv^2 \sin^2 \varphi = 0. \tag{4}$$

Letting $\tau \equiv \tan\theta$ and $\gamma \equiv \cot\varphi$, (3) and (4) may be transformed to

$$(gr^2 - 2hv^2)\tau^2 + 2rv^2\tau + gr^2 = 0,$$

$$gr^2\gamma^2 - 2rv^2\gamma - (2hv^2 - gr^2) = 0.$$

Hence, under the conditions of the problem

$$\tau = \frac{r(v^2 + R)}{2hv^2 - gr^2}, \qquad \gamma = \frac{v^2 + R}{gr},$$

where

$$R \equiv + \sqrt{v^4 + 2ghv^2 - g^2r^2}.$$

Consequently

$$\cos\theta = \frac{2hv^2 - gr^2}{v\sqrt{2(2h^2v^2 - ghr^2 + r^2v^2 + r^2R)}}, \tag{5}$$

$$\cos\varphi = \frac{v^2 + R}{v\sqrt{2(gh + v^2 + R)}}, \tag{6}$$

By considering solid angles, it may be readily seen that the mass which is deposited on the circular area of radius r, corresponding to the cone of half-angle θ, is given by $\frac{1}{2}m(1 - \cos\theta) \equiv m_\theta$. Similarly the mass issuing from the upper or φ cone and falling on the same area equals $\frac{1}{2}m(1 - \cos\varphi) \equiv m_\phi$. The total mass on the area πr^2 is, therefore, $m_\theta + m_\phi \equiv m_r$. The total mass distributed over the ring $2\pi r dr$ is $(dm_r/dr)dr$, so that the surface density is given by

$$\sigma_r = \frac{1}{2\pi r}\left(\frac{dm_\theta}{dr} + \frac{dm_\phi}{dr}\right). \tag{7}$$

Assuming $v^4 + 2ghv^2 - g^2r^2 \neq 0$ the derivatives involved in (7) may be formed by the aid of (5) and (6), so that, in general

$$\sigma_r = \frac{m}{4\pi vR}\left\{\frac{(gr^2 + 2hR)(v^2 + R)^2}{[2(2h^2v^2 - ghr^2 + r^2v^2 + r^2R)]^{\frac{3}{2}}} + \frac{g^2(2gh + v^2 + R)}{[2(gh + v^2 + R)]^{\frac{3}{2}}}\right\}.$$

Special cases:

(*a*) When $R = 0$, that is, when r attains its superior limit $v/g\sqrt{v^2 + 2gh}$ the θ and φ trajectories coincide, the angles θ and φ become supplementary, and $\varphi = \tan^{-1}(1/v\sqrt{v^2 + 2gh})$.

(*b*) Taking $g = 32$ ft./sec.², $h = 5{,}000$ ft., $v = 1{,}024$ ft./sec., the maximum value of r equals $4{,}608\sqrt{66} \doteq 37{,}435.57$ ft. For $r = 0$, $\sigma = 3{,}669 \times 10^{-12}$m. When $2hv^2 - gr^2 = 0$ (*i. e.*, $\theta = \pi/2$) $\sigma = 1{,}612 \times 10^{-13}$ m. When $R = 6{,}604$, $\sigma = 1{,}823 \times 10^{-11}$ m. In the last instance $r \doteq 37{,}435.0001$ ft. If the entire mass were distributed uniformly over the maximum circle the surface density would be approximately $2{,}271 \times 10^{-13}$ m.

MECHANICS.

296. Proposed by C. N. SCHMALL, New York City.

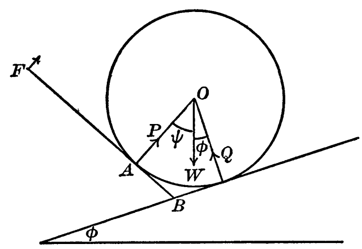
A force F is exerted in moving a horizontal cylinder up an inclined plane by means of a crowbar of length l. If R be the radius of the cylinder, W its weight, φ the inclination of the plane to the horizontal and ψ the inclination of the crowbar to the horizon, show that

$$F = \frac{WR\sin\varphi}{l[1 + \cos(\varphi + \psi)]}.$$

SOLUTION BY A. M. HARDING, University of Arkansas.

The cylinder is in equilibrium under the three forces P, Q, and W. Hence, by Lami's Theorem, $P : Q : W = \sin \varphi : \sin \psi : \sin (\varphi + \psi)$.

$$\therefore \quad P = \frac{W \sin \varphi}{\sin (\varphi + \psi)}, \qquad Q = \frac{W \sin \psi}{\sin (\varphi + \psi)}.$$



From the triangle AOB we obtain

$$AB = R \tan \frac{\varphi + \psi}{2} = R \frac{\sin (\varphi + \psi)}{1 + \cos (\varphi + \psi)}.$$

Taking moments about B we obtain

$$F \times l = P \times AB = \frac{WR \sin \varphi}{1 + \cos (\varphi + \psi)};$$

whence

$$F = \frac{WR \sin \phi}{l[1 + \cos (\phi + \psi)]}.$$

Solved similarly by CLIFFORD N. MILLS.

NUMBER THEORY.

210. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

If a and b are relatively prime and $(a + b)$ is even, then $(a - b)ab(a + b) \equiv 0$, (mod. 24).

SOLUTION BY R. M. MATHEWS, Riverside, California.

The statements a and b relatively prime and $a + b$ even imply

$$a = 2m + 1, \; b = 2n + 1$$

and

$$a + b \equiv 0 \text{ (mod. 2)},$$

$$a - b \equiv 0 \text{ (mod. 2)}.$$

If m and n both odd or both even,

$$a - b \equiv 0 \text{ (mod. 4)}.$$

If m and n be one odd and the other even,

$$(a + b) \equiv 0 \text{ (mod. 4)}.$$

Thus, in any case,

$$(a + b)(a - b) \equiv 0 \pmod{8},$$

a and b each belong to one of the types $3k$, $3k + 1$, $3k + 2$.

If either be $3k$, then

$$ab \equiv 0, \text{ mod. } 3.$$

If both be of type $3k + 1$, or $3k + 2$, then

$$a - b \equiv 0, \text{ mod. } 3.$$

If one be of type $3k + 1$ and the other $3k + 2$, then

$$a + b \equiv 0, \text{ mod. } 3.$$

Thus in any case

$$(a + b)ab(a - b) \equiv 0 \pmod{24}.$$

QUESTIONS AND DISCUSSIONS.

Edited by U. G. Mitchell, University of Kansas.

NEW QUESTIONS.

29. While studying the problem of two equal rough bodies, connected by an inelastic wire, resting on an inclined plane, Professor Clifford N. Mills of South Dakota State College met the following interesting expression. If $1/a$ and $1/a + 1$ are the coefficients of friction, the tension of the wire when the bodies are about to descend becomes a multiple of $1/a[1/a + 1/a + 1]$. This, when simplified, becomes $(2a + 1)/2a(a + 1)$. If $2a + 1$ and $2a(a + 1)$ represent the base and altitude of a right triangle the hypothenuse is $2a^2 + 2a + 1$. Therefore this gives a series of numbers which satisfy the relation $x^2 + y^2 = z^2$, if a is given any value whatsoever. Professor Mills desires to know if this will give all the integers which satisfy the condition that the sum of the squares of two integers equals the square of an integer.

REPLIES.

23. What should be done with the theory of limits in elementary geometry? Should the recommendation of the National Committee of Fifteen be universally adopted? If not, what better disposition of the subject can be made?

Reply by R. M. Matthews, Riverside, California.

In elementary euclidean plane geometry the theory of limits is used at four places. The last of these is for the evaluation of a particular sequence which defines a definite limit π and so is different from the other three which are the theorems fundamental to proportion and the comparison of areas. The first is, "In a circle, central angles are proportional to their intercepted arcs." It depends upon the theorem that equal central angles intercept equal arcs, a relation which is established by congruence. The second is, "A set of parallel lines intercept proportional segments on all transversals." It depends upon the theorem that parallels which intercept equals on any transversal intercept equals on every transversal, a relation also established by congruence. The third is, "Rectangles are proportional to the products of their bases and altitudes"

and its proof is essentially an extension of the theory of measurement implied in the second.

These three propositions can be treated in either of two ways. There is the purely geometric theory of proportion which Euclid developed and which we do not teach. It is not essentially dependent on the idea of ratio or on a complete theory of arithmetic. On the other hand, there is the method of basing geometric proportion on the theory of arithmetic proportion. This latter theory depends on the idea of the ratio of two numbers and connection between arithmetic and geometry is made by defining the ratio of two geometric magnitudes as the ratio of their numerical measures. Accordingly, the whole matter becomes a question of measurement and there would be no difficulty if all magnitudes were "commensurable." This distinction of "commensurable" and "incommensurable" is a survival from Euclid's purely geometric treatment and is needless when a complete system of real numbers is applied to the theory of measurement.

In the measurement of linear segments, the standard distinction is that two segments are commensurable when there exists a segment which can be applied to each an *integral* number of times; otherwise they are incommensurable. Thus, the primary concept is *the number of times one segment is contained in another.* In this primary concept we retain the impression of the simple operation of applying the one segment to the other a step at a time and so argue that "half a time" is absurd. Nevertheless, we practically abandon this primary impression and enlarge our idea of "number of times" when we attach a definite meaning to the statement that segment AB is contained $7\frac{56}{81}$ (for example) times in segment XY. Theoretically we conceive of AB as divided into 81 equal parts and that one of these parts is contained $(7 \times 81 + 56)$ times in XY. The notion of "times contained" has been extended so as to render available all the advantages of the rational number system.

The side and diagonal of a square are declared incommensurable because no segment exists which is contained in both a rational number of times. We need not be confined to rational numbers, however, and the idea of "number of times contained" may be enlarged so as to comprehend every case.

Arithmetically, $\sqrt{2}$ may be defined as the class of all convergent infinite sequences whose limits are the same as the limit of the infinite sequence of rational numbers 1, 1.4, 1.41, 1.414, ...

Geometrically, the side of a square may be applied, in the primary sense, to the diagonal once and a point A_1 determined. A tenth part of the side may be applied 14 times and a point A_2 determined; a hundredth part may be applied 141 times and a point A_3 determined; and so on. This process defines on the diagonal an infinite sequence of points which converges to the end point of the diagonal as a limit. If a different subdivision of the side be used, a different sequence of points would result which, nevertheless, would be equivalent to the first in that it too would converge to the end point of the diagonal.

Thus, on the one hand, there is a class of equivalent convergent sequences of rational numbers, and, on the other hand, a corresponding class of equivalent

convergent sequences of points. Dedekind's axiom is to the effect that to the irrational number defined by the class of numerical sequences corresponds one and only one point, namely, that which is the limit point of the corresponding sequences of points. This, it seems to me, is the adequate enlargement of our idea of "number of times contained" so that we can speak as definitely and rigorously of one segment being contained $\sqrt{2}$ times in another segment as we do when we say that one is contained 1.4 times in another. In brief, *for the theory of measurement in elementary geometry the limit process has been applied once for all in the correlation of real numbers with the points on a line.*

The present difficulties are encountered because we are allowing ourselves to be confined to the number system that Euclid knew—and which was adequate for his method of treating proportion—while trying to treat proportion by arithmetic, which he did not do.

According to the foregoing argument, the results of modern research justify us in regarding every two segments as commensurable and, consequently, all theorems on proportional lines, angles and areas will be proved as they now are in the conventional "commensurable case."

It may be asked, however, what must be taught to high school students for the Dedekind axiom and how a revised proof would run. The usual distinction between commensurable and incommensurable would be replaced by the explicit assumption: *For every two like magnitudes there exists a number that denotes the number of times the one is contained in the other.*

Not only is the equivalent of this axiom at the foundation of analysis but it is the very assumption that seems reasonable to the student. Moreover, we do use it. Let us, therefore, make it explicit.

In secondary school work irrationals are introduced in arithmetic and some use is made of them in the first course in algebra, Since we proceed as if our students had the whole real number system, is it unreasonable to apply it to geometry? Possibly our methods of teaching irrational numbers may need revision, but that is a question for arithmetic and algebra rather than for geometry.

With the foundations for a simple and complete theory of measurement laid in the needed assumption, the proofs for the three important theorems would be simplified. In fact, they would be virtually the same as in the ordinary "commensurable case" except that the numbers used would represent any real numbers instead of merely integers.

The case of limits in the measurement of the circle is different, since in that case the difficulty is not with the limit (whose existence is postulated) but with the setting up of a numerical sequence which will evaluate that limit approximately.

COMMUNICATION.

To the Editor of the Monthly: It seems that a criticism which I made in reviewing Dowling and Turneaure's *Analytic Geometry* in the March number of the Monthly was not very clear, since Professor Dowling states in the May number that the proof which I said was lacking is given on page 58, whereas I

find there no answer to the point which I wished to raise. Inasmuch as the criticism applies also to another recent text (SMITH and GALE: *New Analytic Geometry*) I will ask space for a more detailed statement of my point.

By definition, the equation of the locus of a point satisfying a given condition is an equation in the variables x and y representing coordinates such that, (1) if a point $P(x, y)$ is on the locus it satisfies the equation, and conversely, (2) if a point $P(x, y)$ satisfies the equation it is on the locus.[1]

In deriving the equation of a straight line, Dowling and Turneaure proceed as follows on pages 57, 58:

> Choose any point $P(x, y)$ on the line joining P_1 and P_2 (given points). Then, slope of segment P_1P = slope of segment P_1P_2. But (Art. 11), slope of $P_1P = (y - y_1)/(x - x_1)$, and slope of $P_1P_2 = (y_2 - y_1)/(x_2 - x_1)$. Therefore $(y - y_1)/(x - x_1) = (y_2 - y_1)/(x_2 - x_1)$. (1)

Immediately following this, the authors change this equation into the slope-point form, the slope form, and the intercept form. They then say, "Equations (1), (2), (3), and (4) are all standard forms of the equation of a straight line." My criticism is that they have not shown that any of these equations is the equation of a straight line. They have shown that equation (1) satisfies the first requirement of the definition of the equation of a locus (*i. e.*, if $P(x, y)$ is on the line it satisfies the equation) but they have not proved the "and conversely" part of the definition to be satisfied (*i. e.*, they have not shown that if a point $P(x, y)$ satisfies the equation it lies on the locus). A complete proof of the "and conversely" is as difficult as the part of the proof which they have given, and it should not be omitted.

Following the last statement quoted, the authors name the several standard forms, and then from these equations conclude that, "the equation of a straight line is of first degree in the variables x and y." They then say, "Conversely, it may be shown that any equation of the first degree in the variables x and y is the equation of a straight line." In the proof of this theorem they assume that they have proved that $y = mx + b$ is the equation of a straight line. Obviously then the proof of this "conversely" cannot nullify the criticism I have made above.

Smith and Gale have made the same omission on page 58 of their text.

E. G. MOULTON.

NORTHWESTERN UNIVERSITY.

[1] This is essentially the way in which the definition is given in both texts. See page 55 of the former and page 32 of the latter.

NOTES AND NEWS.

EDITED BY W. D. CAIRNS, Oberlin College.

Professor R. M. BARTON of Lombard College, Galesburg, Ill., has been made dean and acting president of that institution.

Mr. GEORGE RUTLEDGE and Dr. B. B. LIBBY have been appointed instructors in mathematics in the Massachusetts Institute of Technology.

Dr. RAYMOND B. ROBBINS has been appointed to an instructorship in mathematics in the Sheffield Scientific School of Yale University.

Professor E. D. ROE, Jr., of Syracuse University, and Mr. F. E. CARR, of Oberlin College, spent the summer in astronomical work at Yerkes Observatory.

Miss ANNA D. LEWIS, instructor in mathematics and astronomy at Mt. Holyoke College, has been appointed head of the department of mathematics in Kentucky College for Women.

Henry Holt and Company has announced RIETZ, CRATHORNE and TAYLOR'S "School Algebra," also YOUNG and SCHWARTZ'S "Elementary Geometry."

The January–February number of the *Rendiconti del Circolo Matematico di Palermo* contains a biographical account of the work of Professor G. B. Guccia by Professor M. DE FRANCHIS.

The June number of the *Pahasapa Quarterly* of the South Dakota School of Mines pays a tribute to Professor H. L. MCLAURY in recognition of his completion of twenty years as head of the department of mathematics.

Professor ELLERY W. DAVIS, of the University of Nebraska, gave the annual address before the Iowa Academy of Sciences on April 30 on the subject "Uncertainties."

The Open Court for August, 1915, contains an article by Professor F. CAJORI on "The Life of William Oughtred," who contributed to the propagation of mathematical knowledge in England during the early part of the seventeenth century.

Professor C. J. KEYSER, of Columbia University, delivered the commencement address at the University of Oregon in June. He gave also the annual address before the chapter of Sigma Xi of the University of Washington on "Science and Religion, or the Rational and the Super-rational."

In the June number of *School Science and Mathematics* Professor M. O. TRIPP, of Olivet College, enumerates "Some simple applications of elementary algebra to arithmetic," and Mr. H. C. Wright describes the use of mathematical apparatus in the university high school of the University of Chicago.

Mr. W. L. HART will be instructor in mathematics and astronomy at the University of Montana for the first semester of the present year, in place of Professor L. S. Hill, who has accepted an appointment at Princeton University. Mr. Hart will return to Chicago after the holidays to finish his work for the doctorate at the University of Chicago.

Miss OLIVE HAZLETT, who has just finished her work for the doctorate at the University of Chicago, has returned to her home in Boston. She holds a traveling scholarship from the Association of Collegiate Alumnæ, which she had hoped to use in Europe during the coming year, but war conditions may make this impracticable.

An article of interest to teachers of mathematics was published in *School and Society* for May 29, under the title "College Pedagogy," by LOUIS W. ROPER, of Pennsylvania State College. A list is given of 33 qualities of a good college instructor made by twenty college juniors and seniors, each of whom had come in contact with about forty instructors in high school and college.

Dr. S. W. REAVES, professor of mathematics at the University of Oklahoma, is resuming his work after a year spent on sabbatic leave studying in Chicago. He completed work for his doctorate at the close of the summer quarter.

Science for July 2 prints the Halley Lecture for the year 1915 on "Measurements of the distances of the stars," in which the Astronomer Royal, Sir F. W. Dyson, recounts the progress made up to the present in measuring the parallaxes of the nearer fixed stars.

Professor ARTHUR D. PITCHER, of Dartmouth College, has been appointed head of the department of mathematics in Adelbert College, Western Reserve University; and Dr. JOHN M. STETSON has been appointed instructor in mathematics.

The *Mathematical Gazette* for May contains two papers of special interest: "Notes on the Board of Education circular No. 851" (a revision of an earlier circular on the teaching of geometry), by Miss M. J. PARKER, and the first instalment of "The discovery of logarithms by Napier," by Professor H. S. CARSLAW. It will be interesting to compare the latter as it appears with Professor Cajori's "History of the exponential and logarithmic concepts" which appeared in the MONTHLY during the year 1913.

An organization of men who are teaching mathematics in secondary schools of Chicago and vicinity has just started on its second year of activity. They meet monthly for an informal dinner and discussion of live topics of the day in mathematics.

In the thirteenth yearbook of the National Society for the Study of Education appears a paper by H. C. Morrison, superintendent of public instruction for New Hampshire, under the title "Reconstructed mathematics in the high school: the adaptation of instruction to the needs, interests, and capacities of students." This discussion, guided largely by the utilitarian viewpoint, examines critically the extent to which arithmetic, geometry and algebra are needed by different groups of students, *e. g.*, domestic arts, agriculture, mechanic arts, etc., and insists that each shall have its own specially organized mathematics, in charge of specialists in that curriculum.

Mr. W. D. REEVE, formerly instructor in mathematics at the University of Chicago High School, is now in charge of the mathematical courses in the department of education in the University of Minnesota. Mr. Reeve is a graduate of the University of Chicago and has been on the High School staff for several years.

In the last volume of the "Proceedings of the Society for the Promotion of Engineering Education" more than 50 pages are devoted to articles on the teaching of college mathematics and discussion of this subject. The articles are: "Practical mathematics," by Professors FRANKLIN, MACNUTT, and CHARLES; "The proper use of the differential in calculus," by Professor E. V. HUNTINGTON, and "The calculus without symbols," by Professor E. R. HEDRICK. Professor Huntington brings out very clearly, by quotations from texts in calculus, how vague is the presentation of so simple and fundamental concept as that of the differential. The spirited discussion of the first two articles will be found edifying in spots and quite entertaining throughout. No teacher of calculus can afford to miss reading these papers and their discussion.

After the present year the College Entrance Examination Board will conduct all entrance examinations for Yale, Princeton, and Harvard Universities. In view of this greatly increased scope of the Board's work, it is anticipated that 10,000 candidates will be dealt with in the June, 1916, examinations. The readers in mathematics for the present year are as follows:

Algebra.—Professor C. R. MacInnes (chief reader), Princeton; Edward B. Chamberlain, Franklin School, New York; Miss E. B. Cowley, Vassar; C. A. Ewing, Tome School, Port Deposit; Miss Marcia Latham, Hunter; Professor W. R. Longley, Yale; I. F. McCormick, Albany Academy; George W. Mullins, Barnard; C. B. Walsh, Ethical Culture School, New York.

Geometry.—Professor Virgil Snyder (chief reader), Cornell; F. C. Baldy, St. Mark's School, Southboro; G. M. Conwell, Yale; Professor H. N. Davis, Harvard; L. S. Dederick, Princeton; J. G. Estill, Hotchkiss School, Lakeville; J. R. Gardner, Allen-Stevenson School, New York; G. M. Green, Harvard; Professor J. C. Hardy, Williams; Lester E. Lynde, Phillips Academy, Andover; F. J. McMackin, Columbia; E. B. Morrow, Gilman Country School, Roland Park; Professor A. D. Pitcher, Western Reserve; H. W. Reddick, Columbia; H. L. Sweet, Phillips Exeter Academy; L. L. Whitney, Collegiate School, New York.

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THE TEACHING OF MATHEMATICS.

By H. E. SLAUGHT.

Owing to a wide divergence of opinion among college and university men as to the proper interpretation of the term *pedagogy*, and especially in view of the varying notions of that term in the minds of mathematicians, it seems desirable to formulate as definitely as possible what is conceived to be the relation of the MONTHLY to the question of *teaching* in the field of collegiate mathematics.

A certain kind of formal didactics, which at one time prevailed widely in schools for the training of elementary teachers, and which also characterized the earlier attempts at pedagogical training for secondary teachers, has served to alarm many college and university men, whenever the question of college teaching is mentioned, lest the same stilted and formal methods be carried over into the collegiate field. These men have seen some horrible example of a college teacher using what they describe with fine scorn as "high school methods," and thereby sacrificing the higher spirit of the subject to the lower criterion of *so-called pedagogical form*. They say, and with good reason let it be granted, that they would far rather have a young college instructor with good training, fine spirit, and high ideals, who had no teaching experience and no pedagogical training, rather than one whose initiative and individuality is so hedged about with formal rules on teaching and methods of presentation that he becomes a mere machine. And having thus spoken, these people feel that they have said the last word on the teaching of mathematics so far, at least, as the collegiate field is concerned. They think that the teacher is born, not made, that if the young fellow has it in him he will become a good teacher after a certain period of floundering, and if it is not in him, no amount of pedagogical training can make him a good teacher.

There is so large an element of truth in the foregoing argument that many are fully convinced by it. It is true that a bright student from the graduate school may go directly to the teaching of college freshmen and may eventually become proficient at the work, just as it is also true that a student may go directly from the medical school to the independent practice of medicine without the professional training of the clinic and the hospital interneship.

It is abundantly evident to the impartial observer that the teaching of college freshmen and sophomores is not so widely different from the teaching of high school juniors and seniors; and that, on the whole, there is as large a proportion of poor teaching done in the colleges as in the high schools; with the certainty that this proportion will rapidly increase unless remedial measures for the colleges are soon undertaken comparable with those now in operation for the high schools. The logical step forward is a graduate school of education and graduate departments of education which shall give serious and scientific attention to the betterment of college teaching. Meanwhile, so far as mathematics is concerned, the MONTHLY advocates certain general measures which it conceives to be of fundamental and far-reaching importance:

1. **The Stimulation of Individual Initiative.** It is, of course, recognized that in the last analysis good teaching depends upon the initiative of the individual, and to stimulate such initiative is a prime object of the MONTHLY. It is believed that the best way to accomplish this is not by direct reference to the daily routine, nor by dwelling upon pedagogical tricks, but by bringing to the teacher's attention regularly and systematically, in attractive form, the livest discussions possible concerning all phases of work in which he is engaged and in which he should, by virtue of his position, be interested. To this end the MONTHLY has brought to its readers, since its reorganization in 1913, two exceedingly stimulating series of articles by Professor Cajori, one on the History of the logarithmic concept and the other on the History of Zeno's arguments, being phases in the development of the theory of limits, besides half a dozen other equally valuable articles on historical subjects. It has provided no less than thirty articles relating to the teaching of mathematics, the nature of which may be inferred from a few titles taken at random: "Mathematical troubles of the freshman," "Incentives to mathematical activity," "Synthetic projective geometry as an undergraduate study," "On the cultural value of mathematics," "On courses in the history of mathematics," "Determinant formula for the coplanarity of four points," "Some things we wish to know," "What can the colleges do toward improving the teaching of mathematics in the secondary schools?", "Conference periods for college students," etc.

Other lines of personal stimulus have been offered through numerous contributions of general mathematical information, such as the various meetings of mathematicians in this country and in Europe, for example, the "Napier tercentenary celebration," and the "Paris report on calculus in the secondary schools"; through an abundance of shorter notes on matters of mathematical interest, such as comments on various publications, and excerpts from foreign journals in this field, for example, the "Life of Pythagoras" condensed from an exhaustive article by Professor W. W. R. Ball in the *Mathematical Gazette;* and by critical and helpful reviews of new mathematical books in the collegiate and advanced secondary fields.

Finally, in addition to these many forms of stimulus of a more or less indirect nature, the MONTHLY has sought to bring to the individual teacher a direct and

cogent appeal to personal initiative by providing a long list of mathematical papers that involve a minimum of technical treatment and presuppose, for the most part, only the topics usually given in undergraduate courses of study; and a shorter list of papers, involving more technical treatment, for the benefit of those who are ambitious to press on into somewhat higher ranges of mathematical reading. The object of the MONTHLY in selecting these papers is not simply to provide entertainment of a high character, nor even to be content with arousing *passive* interest in this intermediate field, but to *stimulate activity* on the part of college teachers—activity that may be reflected in the class room in the form of renewed enthusiasm, of incisive and critical presentation, of lively and keen interest that is always contagious among the students; and activity that may lead to *production*, to the contribution of similar papers for the benefit of others. If this supreme ambition on the part of the MONTHLY, which is meeting with most encouraging fulfillment in a narrow but gradually widening circle of influence, shall become in anywise general, then we believe that a fundamental and permanent contribution will have been made to the betterment of teaching in the colleges. To this end, every encouragement has been offered to all who will make even a small beginning. The department of Questions and Discussions provides an opportunity for minor contributions of varied character, and the department of Problems makes an appeal to large numbers. The really careful reviewing of a book is an activity that may easily lead to fruitful results. The writing of an acceptable article, historical, critical or technical, may be the beginning of a new epoch for the author; certainly persistence in such activity is capable of transforming a career, as attested by many notable examples.

2. **The Stimulation of Group Organization.** However much depends upon individual initiative and activity, it is certain that group activity is a kind of stimulus that seems to be necessary for the accomplishment of large things. There must be intercommunication and action based upon interchange of ideas. For this reason the MONTHLY has embraced every opportunity to encourage those organizations that have for their object the kind of development above described. For example, the reading club of the California teachers and the new organization among those secondary teachers of New Jersey who wish to discuss mathematical subjects at their meetings rather than mere pedagogical questions. But the aim of the MONTHLY is to see the college teachers of mathematics organized in every state, or even in some smaller groups, for the purpose of mutual assistance along all lines of interest in the collegiate field. A number of such groups are already under consideration, and one is being formally organized in the state of Kansas as this issue goes to press. The discussion at this meeting is on question 27 in the September MONTHLY, namely,

> "A certain college wishes to offer twelve hours of mathematics beyond the usual courses in analytic geometry and calculus. Considering only the needs of students intending to specialize in pure mathematics, what courses should make up the twelve hours offered?"

The MONTHLY congratulates these Kansas teachers and commends their example to all other such groups of college teachers throughout the country. With the

attention of college men fixed upon the varied and special problems of college work in mathematics, and with many group organizations for interchange of ideas, a new era will be inaugurated in this field.

3. **The Stimulation of National Organization.** The most natural climax of wide group organization is a national organization, such as was referred to in the October issue, and conversely, such a national organization will react on the formation of smaller groups and both will provide a far-reaching stimulus to individual activity. The MONTHLY would have welcomed the incorporation of such an organization within the American Mathematical Society, but since this is not to be, we look forward with high hopes and great enthusiasm to the organization of a new national society, and with more than four hundred charter members, we see no reason why commendable things should not be accomplished through this movement for the cause of mathematics in America.

HISTORY OF ZENO'S ARGUMENTS ON MOTION:

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

By FLORIAN CAJORI Colorado College.

X.

E. POST-CANTORION DISSENSIONS (Concluded).

With the advent of the new century, discussion on Zeno began to quiet down in France. We note only two articles. In 1907 O. Hamelin wrote on the "Arrow," but the interest of his article centers in what constitutes the most probable renderings of the Aristotelian text.[1] In 1909 a novel attempt to solve Zeno's puzzles was made by Dunan[2] in an article in which he retracts what he said on this subject in a pamphlet of 1884.[3]

He believes that the difficulties vanish, on the recognition that motion takes place through a space, one and indivisible, without succession and parts. He admits that such a proposition raises considerable difficulty, which cannot be removed except by long and elaborate metaphysics, of which he gives in his article only a bare sketch.

No less radical is the position of Henri Bergson. He holds that philosophy must get back to reality itself. Reality is supplied by intuition. Pure intuition, external or internal, is that of undivided continuity. Every movement, in as much as it is a passage from rest to rest, is in fact absolutely indivisible. Sight perceives the movement in the form of a line which is traversed, and this line, like all space, may be indefinitely divided. We must not confound the data of

[1] *L'année philosophique* de F. Pillon, Paris, 1907, pp. 39–44.

[2] "Zénon d'Élée et le Nativism" in *Annales de Philosophie Chrétienne*, 1909.

[3] *Les arguments de Zénon d'Élée contre le mouvement*, Nantes, 1884.

the senses, which perceive the movement as an undivided whole, with the artifice of the mind which divides into parts the path traversed. Says Bergson:[1]

"You substitute the path for the journey, and because the journey is subtended by the path you think that the two coincide. But how should a *progress* coincide with a *thing*, a movement with an immobility? . . . And from the fact that this line is divisible into parts and that it ends in points, we cannot conclude either that the corresponding duration is composed of separate parts or that it is limited by instants. The arguments of Zeno of Elea have no other origin than this illusion. They all consist in making time and movement coincide with the line which underlies them, in attributing to them the same subdivisions as to the line, in short in treating them like that line. In this confusion Zeno was encouraged by common sense, which usually carries over to the movement the properties of its trajectory, and also by language, which always translates movement and duration in terms of space. . . . But the philosopher who reasons upon the inner nature of movement is bound to restore to it the mobility which is its essence, and this is what Zeno omits to do. By the first argument (the Dichotomy) he supposes the moving body to be at rest, and then considers nothing but the stages, infinite in number, that are along the line to be traversed: we cannot imagine, he says, how the body could ever get through the interval between them. But in this way he merely proves that it is impossible to construct, *à priori*, movement with immobilities, a thing no man ever doubted. The sole question is whether, movement being posited as a fact, there is a sort of retrospective absurdity in assuming that an infinite number of points has been passed through. But at this we need not wonder, since movement is an undivided fact, or a series of undivided facts, whereas the trajectory is infinitely divisible. In the second argument (the Achilles) movement is indeed given, it is even attributed to two moving bodies, but, always by the same error, there is an assumption that their movement coincides with their path, and that we may divide it, like the path itself, in any way we please. Then, instead of recognizing that the tortoise has the pace of a tortoise and Achilles the pace of Achilles, so that after a certain number of these indivisible acts or bounds Achilles will have outrun the tortoise, the contention is that we may disarticulate as we will the movement of Achilles and, as we will also, the movement of the tortoise: thus reconstructing both in an arbitrary way, according to a law of our own which may be incompatible with the real conditions of mobility. The same fallacy appears, yet more evident, in the third argument (the Arrow) which consists in the conclusion that, because it is possible to distinguish points on the path of a moving body, we have the right to distinguish indivisible moments in the duration of its movement. But the most instructive of Zeno's arguments is perhaps the fourth (the Stadium) which has, we believe, been unjustly disdained, and of which the absurdity is more manifest only because the postulate masked in the three others is here frankly displayed. Without entering on a discussion which would here be out of place, we will content ourselves with observing that motion, as given to spontaneous perception, is a fact which is quite clear, and that the difficulties and contradictions pointed out by the Eleatic school concern far less the living movement itself than a dead and artificial reorganization of movement by the mind."

Bergson discusses the "Arrow" more fully in his *L'Evolution creatrive*, 1907, where he refers[2] to the absurdity of regarding movement as made up of immobilities. He says:

"Philosophy perceived this as soon as it opened its eyes. The arguments of Zeno of Elea although formulated with a different intention, have no other meaning. . . . Motionless in each point of its course, it is motionless during all the time of its moving. Yes, if we suppose that the arrow can ever *be* in a point of its course. Yes again, if the arrow, which is moving, ever coincides with a position, which is motionless. But the arrow never *is* in any point of its course. The most that we can say is that it might be there, in this sense, that it passes there and might stop there. . . . You fix a point C in the interval passed, and say that at a certain moment the arrow was at C. If it had been there it would have been stopped there, and you would no longer have had a flight from A to B, but *two* flights, one from A to C and the other from C to B, with an interval of rest. A single movement is entirely, by the hypothesis, a movement between two stops; if there are intermediate stops, it is no longer a single movement."

[1] H. Bergson, *Matter and Memory*, transl. by Nancy M. Paul and W. Scott Palmer, London, 1911, pp. 248, 250–253. The first French edition appeared in 1896.

[2] H. Bergson, *Creative Evolution*, transl. by A. Mitchell, London, 1911, pp. 325–327.

There have been many dissussions of Bergson. One writer endeavors to point out his errors by returning to the continuums of Artistotle and Thomas Aquinas.[1] Most pertinent to our topic are the criticisms by Bertrand Russell, of Cambridge, England, which are displayed by the following quotations:[2]

". . . it will be said, the arrow is where it is at any one moment, but at another moment it is somewhere else, and this is just what constitutes motion. Certain difficulties, it is true, arise out of the continuity of motion, if we insist upon assuming that motion is also discontinuous. These difficulties, thus obtained, have long been part of the stock-in-trade of philosophers. But if, with the mathematicians, we avoid the assumption that motion is also discontinuous, we shall not fall into the philosopher's difficulties. A cinematograph in which there are an infinite number of films, and in which there is never a *next* film because an infinite number come between any two, will perfectly represent a continuous motion. Wherein, then, lies the force of Zeno's argument? . . . Zeno assumes, tacitly, the essence of the Bergsonian theory of change. That is to say, he assumes that when a thing is in process of continuous change, even if it is only change of position, there must be in the thing some internal *state* of change. The thing must, at each instant, be intrinsically different from what it would be if it were not changing. He then points out that at each instant the arrow simply is where it is, just as it would be if it were at rest. Hence he concludes that there can be no such thing as a *state* of motion, and therefore, adhering to the view that a state of motion is essential to motion, he infers that there can be no motion and that the arrow is always at rest. Zeno's argument, therefore, though it does not touch the mathematical account of change, does, *prima facie*, refute a view of change which is not unlike M. Bergson's. How, then, does M. Bergson meet Zeno's argument? He meets it by denying that the arrow is ever anywhere. After stating Zeno's argument, he replies: 'Yes, if we suppose that the arrow can ever *be* in a point of its course. Yes again, if the arrow, which is moving, ever coincides with a position, which is motionless. But the arrow never *is* in any point of its course.' (C. E., p. 325.) This reply to Zeno, or a closely similar one concerning Achilles and the Tortoise, occurs in all his three books. Bergson's view plainly, is paradoxical; whether it be *possible*, is a question which demands a discussion of his view of duration. His only argument in its favor is the statement that the mathematical view of change 'implies the absurd proposition that movement is made of immobilities.' (C. E., p. 325.) But the apparent absurdity of this view is merely due to the verbal form in which he has stated it, and vanishes as soon as we realize that motion implies relations. A friendship, for example, is made out of people who are friends, but not out of friendships. . . . So a motion is made out of what is moving, but not out of motions. It expresses the fact that a thing may be in different places at different times, and that the places may still be different however near together the times may be. Bergson's argument against the mathematical view of motion, therefore, reduces itself, in the last analysis, to a mere play upon words." . . .

"Mathematics conceives change, even continuous change, as constituted by a series of states; Bergson, on the contrary, contends that no series of states can represent what is continuous, and that in change a thing is never in any state at all." . . .

"One of the bad effects of an anti-intellectual philosophy, such as that of Bergson, is that it thrives upon the errors and confusions of the intellect. Hence it is led to prefer bad thinking to good, to declare every momentary difficulty insoluble, and to regard every foolish mistake as revealing the bankruptcy of intellect and the triumph of intuition. . . . As regards mathematics, he has deliberately preferred traditional errors in interpretation to the more modern views which have prevailed among mathematicians for the last half century."

Thus it is seen that among recent French philosophers the Cantor continuum has been neglected and no satisfactory substitute has been advanced.

A treatment of the "Achilles" altogether different from that hitherto given by either philosophers or mathematicians is given by Russell.[3] After explaining infinite number and the modern continuum, he says in the *International Monthly:*

[1] T. J. Gerrard, *Bergson, an Exposition and Criticism*, London and Edinburgh, 1913, pp. 23 ff.

[2] B. Russell, "The Philosophy of Bergson," *The Monist*, July, 1912. Russell's references (C. E.) are to the English translations of Bergson's *Creative Evolution*.

[3] See B. Russell, *Principles of Mathematics*, 1902; "Recent work on the Principles of Mathematics" in the *International Monthly*, Vol. IV, 1901, pp. 83–101.

"We can now understand why Zeno believed that Achilles cannot overtake the tortoise and why as a matter of fact he can overtake it. We shall see that all the people who disagreed with Zeno had no right to do so, because they all accepted premises from which his conclusion followed. . . . Then he [Achilles] will never reach the tortoise. For at every moment the tortoise is somewhere, and Achilles is somewhere; and neither is ever twice in the same place while the race is going on. Thus the tortoise goes to just as many places as Achilles does, because each is in one place at one moment, and in another place in another moment, and in another at any other moment. But if Achilles were to catch up with the tortoise, the places where the tortoise should have been, would be only part of the places, where Achilles would have been. Here, we must suppose, Zeno appealed to the maxim that the whole has more terms than the part. Thus if Achilles were to overtake the tortoise, he would have been in more places than the tortoise; but we saw that he must, in any period, be in exactly as many places as the tortoise. Hence we infer that he can never catch the tortoise. This argument is strictly correct, if we allow the axiom that the whole has more terms than the part. As the conclusion is absurd, the axiom must be rejected, and then all goes well. But there is no good word to be said for the philosophers of the past two thousand years and more, who have all allowed the axiom and denied the conclusion. The retention of this axiom leads to absolute contradictions, while its rejection leads only to oddities."

The conjectures which Russell makes on the history of the "Achilles" are, in the main, without foundation. There is no historical evidence for believing that Zeno based the "Achilles" on the doctrine that the whole is greater than any of its parts. Aristotle bases Zeno's argument on the assertion that a line or distance cannot be reduced by any process of successive division to elements that are mathematical points. Russell's version of the paradox is what Zeno might have said, but did not actually say. It is far simpler to explain than is that of Zeno. Assent can be readily secured to the fact that, in infinite aggregates, the whole is not greater than certain of its parts.[1]

That Russell's argument, though correct in itself, does not meet the exact difficulty experienced by many persons, is brought out by C. D. Broad,[2] who points out that the "Achilles" rests on the false assumption that "what is beyond every one of an infinite series of points, must be infinitely beyond the first point of the series." Broad considers it important even at this time to settle this controversy, "because it and Zeno's other paradoxes have become the happy hunting-ground of Bergsonians and like contemners of the human intellect." What makes infinite divisibility a stumbling block to so many is the fact that appeal is made to sensory intuition and imagination—the very faculty of the mind which proves itself unable to cope with the problem. But our powers of analysis penetrate realms of thought beyond the reach of the imagination, and it is in that territory that the arguments of Zeno are made to surrender their mysteries.

B. Russell took great interest also in the "Arrow." In the *International Review* he remarked:

[1] In this connection a story told by De Morgan may be of interest. He relates "a tradition of a Cambridge professor who was once asked in a mathematical discussion, 'I suppose you will admit that the whole is greater than its part?' and who answered, "not I, until I see what use you are going to make of it.' " The danger of unintended implications is illustrated by an author who remarked that Gibbon always had a copy of Horace in his pocket and often in his hand, from which it would seem to follow that Gibbon's hand was sometimes in his pocket.

[2] *Mind*, Vol. 38, 1913, pp. 318, 319.

"Weierstrass, by strictly banishing from mathematics the use of infinitesimals, has at last shown that we live in an unchanging world, and that the arrow in its flight is truly at rest. Zeno's only error lay in inferring (if he did infer) that, because there is no change, therefore the world is in the same state at any one time as at any other. . . . Weierstrass has been able, by embodying his views in mathematics, where familiarity with truth eliminates the vulgar prejudices of common sense, to invest Zeno's paradoxes with the respectable air of platitudes." Elsewhere Russell expresses much the same idea by the statement that "a variable does not vary."[1]

That a variable cannot reach its limit is still widely held. In 1907 R. B. Haldane presented this as the teaching of mathematics, in a presidential address to the Aristotelian Society, entitled "The Methods of Modern Logic and the Conception of Infinity." In a review of this address, B. Russell says[2] that this property "belongs to limits of a certain particular sort," which constitute "an extremely special case, not realized in most of the series in which limits exist."

The creation of the theory of sets and of the Cantor continuum lead to modified definitions of the limit. In this theory the concept of a limit was divorced from the idea of quantity and measurement. The question whether the variable reaches its limit or not is ignored as being of no interest. Whether it reaches its limit or not depends upon the nature of the variation in a particular case; the sequence of values may include the limit, or it may not. The *limiting point* of a set of points is one for which every interval, however small, containing the limiting point, encloses a point of the set, distinct from the limiting point itself. A *limit* is merely the arithmetical equivalent of the limiting point in geometry. The introduction of a transfinite number as a limit has carried with it still further modification of the idea of a limit. Small intervals do not fit here. Says Bertrand Russell: "If we consider the whole series of integers, finite and infinite, arranged in order of magnitude, then the class of finite integers, considered as part of this series, has an upper limit, namely the smallest of the infinite integers (which is the number of finite integers)." Here there is no "negligible difference" between variable and limit; "the difference between the finite integers and their limit remains constant and infinite." Again he says: "A limit must not be conceived as something to which the successive terms of the class approach indefinitely near; they may all be at an infinite distance from the limit, or at a distance which remains permanently greater than some given finite distance; or the series concerned may be one in which there is no such thing as distance or difference." His definition of a limit is as follows: "Given any series, and a class α of terms belonging to the series, a term x belonging to the series is called the *upper limit* of α if every term of α precedes x, and every term of the series which precedes x precedes some member of α." He gives a similar definition for *lower limit*.[3] It is to be observed that the modern definitions of a limit are free from the concept of the old-time infinitesimal.

As now we pause and look backward, we see that a full and logically correct explanation of Zeno's arguments on motion has been given by the philosophers of mathematics. Looking about us, we see that the question is still regarded as

[1] B. Russell, "Mr. Haldane on Infinity," *Mind*, Vol. 33, London, 1908, p. 240.
[2] *Mind*, Vol. 33, 1908, p. 239.
[3] B. Russell, "Mr. Haldane on Infinity," *Mind*, Vol. 33, London, 1908, pp. 240, 241.

being in an unsettled condition. Philosophers whose intellectual interests are remote from mathematics are taking little interest in the linear continuum as created by the school of Georg Cantor. Nor do they offer a satisfactory substitute. The main difficulty is not primarily one of logic; it is one of postulates or assumptions. What assumptions are reasonable and useful? On this point there is disagreement. Cantor and his followers are willing to assume a continuum which transcends sensuous intuition. Others are not willing to do so. Hence the divergence. In the Koran there is a story that, after the creation of Adam, the angels were commanded to make him due reverence. But the chief of the angels refused, saying: "Far be it from me a pure spirit to worship a creature of clay." For this refusal he was shut out from Paradise. The doom of that chief, so far as the mathematical paradise is concerned, awaits those who refuse to examine with proper care the massive creation by our great mathematicians, without which the tiniest quiver of a leaf on a tree remains incomprehensible.[1]

TABLE OF CONTENTS.

MATHEMATICAL MEETINGS IN CALIFORNIA.

I. THE TWENTY-SECOND SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The American Mathematical Society met for its twenty-second summer meeting as announced by the Society, on August 3, 1915, at the University of California in Berkeley. The first meeting was in conjunction with Section A of the American Association for the Advancement of Science, on Tuesday morning. Professor Keyser, of Columbia University, delivered an address on the human significance of mathematics, and Director Hale, of the Mount Wilson Solar Observatory, delivered an address on the work of a modern observatory. The attendance was very large and included members of the American Mathematical

[1] Since the completion of this article there has appeared Bertrand Russell's *Our Knowledge of the External World as a Field for Scientific Method in Philosophy*, Open Court Company, 1914, in which much attention is given to Zeno's arguments. An article on Zeno by Philip E. B. Jourdain will soon appear in *Mind*.

Society and the American Astronomical Society as well as members of the Association at large. At least 250 persons were present. Following this session the two societies enjoyed a luncheon at the University Faculty Club as the guests of Professors Leuschner, Haskell and E. B. Lewis.

The first separate session of the Society was held Tuesday afternoon at the University of California, where papers were presented by Professor L. E. DICKSON, University of Chicago; Professor C. J. DE LA VALLÉE POUSSIN, University of Louvain; Mr. A. R. SCHWEITZER, Chicago, Ill.; Dr. NATHAN ALTSHILLER, University of Colorado; Professor L. J. RICHARDSON, University of California; Dr. DUNHAM JACKSON, Harvard University; Dr. W. W. KÜSTERMANN, University of Michigan; Professor G. A. MILLER, University of Illinois; Professor H. S. WHITE, Vassar College; and Professor M. W. HASKELL, University of California.

The members present were largely from the Pacific coast, although a few from the eastern states were present. The lists of those present will appear as usual in the BULLETIN of the American Mathematical Society.

On Wednesday the Society met at Stanford University and papers were presented by Dr. B. A. BERNSTEIN, University of California; Dr. C. A. FISCHER, Columbia University; Mr. A. R. WILLIAMS, and Professor L. M. HOSKINS, Stanford University. Since this completed the program the other sessions announced were not necessary.

Returning to Berkeley Wednesday evening the society enjoyed dinner again in conjunction with the American Astronomical Society at the Hotel Oakland in Oakland.

On Friday about 50 members of the two societies took a very delightful trip to the Lick Observatory on Mount Hamilton, which was reached by automobiles from San José, and after remaining overnight in San José, enjoyed the hospitality of Mrs. Hearst on the return trip to San Francisco.

A more detailed account of this meeting will of course appear in the BULLETIN of the American Mathematical Society. To those easterners who were present the occasion was particularly delightful and was a revelation of the possibilities of the Coast both as a mathematical center and as an enjoyable meeting place.

II. OTHER MEETINGS AT CALIFORNIA.

In addition to the meeting of the American Mathematical Society in California, a large number of other meetings were held in connection with the exposition, the meetings themselves being largely in Berkeley, the seat of the University of California, or in Oakland, which immediately adjoins Berkeley.

Among the meetings of particular interest to the readers of the MONTHLY would be the American Association for the Advancement of Science, which was held during the entire week of August 1–7, and which will be reported in detail in the *Proceedings* of the Association. The attendance at this meeting was very large, and the members of the various component societies thought that the occasion was well chosen for a meeting. This is the more remarkable on account of the fact that the meeting was specially called, since the Association now has its regular meetings during the Christmas holidays.

Another gathering of peculiar interest was the meeting of the Association of American Agricultural Colleges and Experiment Stations, August 11–13, at Berkeley. The name of this association may seem to indicate that it is interesting only to agriculturalists but attention should be called to the fact that the Association includes all of the so-called Land-Grant colleges, which receive aid under various national acts from the United States government. Thus many state universities were represented at this meeting, and such institutions as the Massachusetts Institute of Technology. A subsidiary association of considerable importance to those interested in mathematics was the Land Grant College Engineering Association, which met at the same time. To those who are not aware of the purposes of these two associations it may be pointed out that one very important project before them at present is to secure government aid for the establishment of engineering experiment stations in the various land-grant colleges.

Finally, the meeting of the National Education Association held in Oakland, August 16–28, will be of interest to those who have to do with secondary education either directly or indirectly. No full account of this meeting is possible and we must content ourselves with the statement that it was a very large meeting and that a full report of it will be published by the Association itself. One discussion that occurred there which will be of interest to all who read the MONTHLY was concerned with the establishment of six-year high schools—a movement which would seem to be well under way and to promise great changes and considerable success in the near future.

A large number of other meetings of educational importance were held, including for example the Association of American Universities and the Association of American State Universities.

On the whole the meetings held this summer in California will certainly take rank as one of the most important groups of meetings which have ever been held in the United States.

HISTORY OF MATHEMATICS.

By G. A. MILLER, University of Illinois.

In February, 1640, Descartes wrote as follows: "I am accustomed to distinguish two things in the mathematics, the history and the science. *By the history I mean whatever is already discovered, and is committed to books.* And by the science, the skill of resolving all questions, and thence by investigating by our own industry whatever may be discovered in that science by human ingenuity. He who possesses this faculty has but little need of other assistance, and may therefore be properly called self-sufficient. Now it is much to be wished that this mathematical history, which lies scattered through many volumes, and is not yet entire and complete, were to be all collected into one book."[1]

[1] *The Philosophical Transactions of the Royal Society of London* (Abridged), Vol. 2, 1809, p. 533.

Descartes's definition of history of mathematics, implied by the sentence of the preceding quotation which was italicized by the present writer, may at first surprise many of those who have acquired their idea of the meaning of the term history of mathematics from the modern books bearing this title. A second thought will doubtless reveal difficulties in the way of defining clearly what we really mean by this term. Are the mathematical developments of the first decade and a half of the present century a part of the history of mathematics? If this is the case, should we not begin the study of the history of mathematics by the study of this period?

To direct attention to the feasibility of beginning a study of the history of mathematics near its border which is closest to us, it may be interesting to recall a few mathematical activities which were inaugurated since the beginning of the present century. The one which is probably of most interest to the average teacher is the work done under the general direction of the *International Commission on the Teaching of Mathematics,* which was created during the fourth international congress held at Rome in April, 1908. At first it was intended that this commission should confine its work to secondary mathematics, but it soon appeared desirable to include all mathematical instruction in the scope of its investigators.[1] The central committee is now composed of seven men representing seven leading countries as follows: F. Klein, Germany; G. Greenhill, England; D. E. Smith, United States; H. Fehr, Switzerland; G. Castelnuovo, Italy; E. Czuber, Austria; and J. Hadamard, France.

The magnitude of the work accomplished during the first six years of the existence of this Commission is partly exhibited by the fact that over ten thousand pages of reports prepared under its general direction were issued during this period, according to a statement made at the meeting held at Paris in April, 1914. These reports relate mainly to the materials and methods of teaching mathematics in sixteen different countries, including all those which lead in scientific activities, and they constitute a most valuable addition to the literature on the development of mathematics in various modern countries. Several additional reports have appeared since the given meeting, and it seems probable that the work inaugurated by this Commission will bear fruit for a long time to come.

Another large mathematical undertaking inaugurated during the first decade and a half of the present century is the publication of the great mathematical encyclopedia, entitled *Encyclopédie des Sciences Mathématiques,* the first part of which was published in 1904. More than thirty large volumes of this great encyclopedia have been planned and over five thousand pages have already been published. Unfortunately the great European war has greatly delayed this publication, which promises to become the largest and most valuable mathematical encyclopedia that has ever been written.

A considerable number of mathematical periodicals have been started during the period under consideration. About the beginning of this period the *Trans-*

[1] J. W. A. Young gave an account of the work of this Commission during its first four years, in this MONTHLY, Vol. 19 (1912), p. 161.

actions of the American Mathematical Society was started and it has enjoyed the support of leading American mathematicians from the start. The high standards and careful editorship of this periodical have been of the greatest value in the development of research activity of a high order in our midst. Quite recently an important general scientific journal, devoting considerable attention to mathematics, was started in our country, viz., the *Proceedings of the National Academy of Sciences*. The first number appeared in January of the present year, and the periodical at once received the support of leading investigators in the various scientific fields.

In May, 1911, there was started a new Spanish mathematical journal, entitled *Revista de la Sociedad Matematica Española,* which is of special interest to Americans in view of the fact that the Spanish language is used in large parts of our continent. This periodical is the official organ of the national mathematical society of Spain, which was organized about a month before the periodical was started. It is to be hoped that this society and its journal will do much to organize the mathematical work among the people using the Spanish language, especially since these people have not taken an active part in the development of mathematics during recent centuries, having remained far behind the Italians in this regard.

Another important mathematical periodical entitled *Tôhoku Mathematical Journal* was started in 1911 at Sendai, Japan. It invited from the beginning contributions in English, French, German, Italian, and Japanese, but most of its articles thus far have been in English. This is the first journal devoted mainly to modern advanced mathematics which has been published in Japan, and its international character should do much to advance the interests of higher mathematics in that country.

In view of the fact that Asia took practically no part in the development of mathematics in modern times up to the beginning of the twentieth century, it is of interest to note another thriving mathematical journal started in Asia during the period under consideration. This periodical is entitled *The Journal of the Indian Mathematical Society,* and was started at Madras, India, in February, 1909. The fact that it is the official organ of a society founded in 1907 for the advancement of mathematical study and research in India makes it the more interesting and increases its opportunities for usefulness. Another Indian mathematical periodical was started in 1909, under the title *Bulletin of the Calcutta Mathematical Society,* but only four numbers of this journal, which was to be a quarterly, have been issued thus far.

One of the most noteworthy features of the period under consideration is the rapidly increasing interest in questions relating to the teaching of mathematics, and the great success of the International Commission on the Teaching of Mathematics was doubtless largely due to the fact that the time was ripe for vigorous advances along this line. Reforms of various kinds and of far-reaching significance have received widespread attention. A considerable number of new journals devoted mainly to methods of teaching were established during the

decade and a half under consideration. In our own country *School Science and Mathematics* and *The Mathematics Teacher* may serve as illustrations. The former of these is a continuation of *School Science*, the first number of which appeared in March, 1901; while the latter began to appear as a quarterly in September, 1908.

Perhaps no other undertaking started during the period under consideration exhibits the spirit of this period so clearly as the commencement of the publication of the collected works of the great Swiss mathematician, Leonard Euler. Various efforts had been made earlier to publish these very extensive works but these efforts failed on account of the large amount of money required for the publication. In September, 1909, the Swiss Society of Naturalists, having received through national and international subscriptions and through donations about one hundred thousand dollars for this purpose, announced that it would undertake this great publication. Two years later the first volume was published, and several other volumes followed in rapid succession, but it soon appeared that, in view of the discovery of new MSS., the entire publication would cost nearly twice as much as the original estimate, and that it would fill more than forty-five large volumes. This led to the formation of a unique international mathematical society, called the *Leonard Euler-Gesellschaft*, whose main object is to aid this publication.

The developments which have been noted are of a general nature and they constitute merely evidences of the fact that there was real mathematical growth during the period under consideration. It would be of more interest to consider some of the mathematical advances themselves, but these are too numerous and too extensive to be described in a brief article. If it is observed that the *Jahrbuch über die Fortschritte der Mathematik* fills annually a volume of over a thousand pages in giving titles and brief reviews of the new literature on mathematics, it results that for the decade and a half under consideration it would require more than fifteen thousand pages to present even such a limited consideration of the mathematical progress as is contained in the given review. Hence it is clear that a history of the development of mathematics during the first decade and a half of the twentieth century might well fill many volumes.

The main object of the present article is to raise the question whether such books on the history of mathematics as Ball, Cajori, Cantor, etc., are not apt to convey a very incorrect notion of what the history of mathematics really is, and of what developments should be embodied in a first course on this subject. In fact, Cantor's *Vorlesungen über Geschichte der Mathematik* are confined to the developments which preceded the nineteenth century, and hence they do not touch the period in which most of our present mathematical literature originated. Possibly some of the other books could be used to the best advantage as text-books by beginning near the end and working forward, but a really inspiring course in the history of mathematics would probably have to be based largely upon the literature contained in the recent journals.

A course in the history of mathematics should not tend to create a veneration

of the past at the expense of an appreciation of the present. It would be better to be ignorant of the Pythagorean school than to be ignorant of the modern mathematical schools. It would be better to be ignorant of the Greek Eleatic school than to be ignorant of some of the fundamental results in the modern theory of aggregates. It would be better to be able to name with some intelligence ten of the most eminent living mathematicians than to be able to name that number of ancient Arabians who helped to preserve and to transmit the mathematical lore of the ancient Greeks.

One difficulty about beginning a course in the history of mathematics with the developments of the last ten or fifteen years is that the new theorems and theories have not yet established their true value in the permanent literature. Mathematical fashions are changeable, but mathematical worth is permanent. Recent mathematical history is apt to be affected by the fashions while the older mathematical history is based upon established permanent values. These objections do not apply so strongly to a study of the various present activities which were inaugurated with a view to furthering mathematical interests, and these activities should be well understood by all those who teach mathematics.

It is also true that it is somewhat more difficult to keep in touch with the recent developments than it is to study once for all some accounts of the older developments. This difficulty has been greatly reduced in recent years by the increase in the periodical literature and by the various aids to make rapid surveys of the main advances in various fields. There is less and less excuse for living entirely in the mathematical past. The growing dynamic elements of mathematics naturally appeal to a large number, especially to the younger people, and these elements are best understood if they are observed in their natural surroundings and in real life.

The given quotation from Descartes seems to imply that he thought that all the mathematical history of his day could have been collected into *one* book. We noted above that the large French mathematical encyclopedia is expected to fill more than thirty volumes, and this will not include all the known mathematical results. If it is observed that more than two thousand original mathematical articles are published every year, it is clear that mathematical history is growing more rapidly than one man could write it. Hence such a thing as a complete mathematical history seems out of question. It is, however desirable to know something about this history, and especially about that part which lies closest to us.

It is of interest to inquire what distinguishes the history of mathematics from other mathematical writings, in case we assume that there is a difference, as is commonly done. According to Descartes's view practically all the literature which appears in our better journals of mathematics should be regarded as historical, while many others would probably be inclined to contend that only a small part of this literature should be classed with history. All might agree on the statement that every mathematical advance is making mathematical history, but some would probably hold that this advance would become history through

some process of maturing, while the process itself might be vague and hence undefinable. Judging from the writings which are now commonly classed under mathematical history it would appear that the chronological element was generally considered an essential element in a direct historical paper. In an indirect historical paper,—for instance, one discussing how mathematical history should be written—this element would not need to be present.

Next in importance to the chronological element in the usual historical presentation of a mathematical subject is the human element. Even the mere names of those who have enriched mathematical thought by pointing out logical steps leading to views of unusual beauty or to regions of unusual fruitfulness serve to establish a sense of comradeship. This sense is intensified by more or less complete biographical notices. It should, however, be observed that these names and these biographical sketches serve other useful purposes. The names of great mathematicians may be used to unify varied mathematical results, while the biographical sketches serve to unify mathematical knowledge and knowledge relating to other lines of thought.

While these considerations may throw some light on the term history of mathematics they are not intended to convey the idea that this term could be defined in a perfectly satisfactory manner. Like some other useful mathematical terms, the term history of mathematics will probably always remain without a real definition. While one may not know where mathematical history begins, yet there are some writings which all agree to classify under history and others which few would classify under this heading. On the other hand, there seems to be a strong tendency towards increasing the historical element in modern mathematical writings so that the writings which are zero per cent. history are becoming less and less common. It is to be hoped that the mathematical teaching which devotes zero per cent. of the time to history, and the mathematical history which includes zero per cent. of the present-day mathematical activities will also become less and less common.

ON THE CIRCLES OF APOLLONIUS.

(Concluded)

By NATHAN ALTSHILLER, University of Colorado.

(By some unaccountable oversight this last page of DR. ALTSHILLER's article was omitted from the October issue. EDITORS.)

12. The three pairs of points A, A'; B, B'; C, C' of ω are perspective from K; they belong therefore to the same involution on ω, K and l being respectively the pole and the axis of the involution[1] (9A). Hence the couples of lines AB, $A'B'$; BC, $B'C'$; AC, $A'C'$ meet on l. Now, C_3, C_1, C_2 are the points common to l and the lines AB, BC, AC respectively. Hence:

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[1] Russell, *loc. cit.*, pp. 217, 218.

The lines $A'B'$, $B'C'$, $A'C'$ meet the lines AB, BC, AC respectively in the centers of the Apollonian Circles.

13. The points A, B, A', B' are the vertices of a complete quadrangle inscribed in ω. The polars s_3 and l, with respect to ω (8, 9), of the diagonal points $C_3 \equiv (AB, A'B')$ (12) and $K \equiv (AA', BB')$ (7) intersect in the third diagonal point[1] $(AB', A'B)$, *i. e.*, this point is identical with P_3 (11). Similarly for the poles P_1 and P_2. Hence:

The pairs of lines AB', $A'B$; BC' $B'C$; AC', $A'C$ meet respectively in the poles P_3, P_1, P_2 of the Brocard diameter with regard to the Apollonian circles.

14. The triangles KP_1C_1, KP_2C_2, KP_3C_3, are self-conjugate with respect to ω (13), and therefore:

The center of an Apollonian circle and the pole, with regard to this circle, of the Brocard diameter are conjugate points with respect to the circumcircle.

15. The involution of conjugate points P_1, C_1; P_2, C_2; P_3, C_3 with respect to ω (14) is the section of the orthogonal involution of rays having its center at I; the rays joining I to the two points of intersection of 1 with any one of the Apollonian Circles are conjugate in this involution. Consequently:

The points of intersection of the Lemoine line with an Apollonian circle are conjugate with respect to the circumcircle.

16. Lemma. *If two tangents to two circles meet on the radical axis, the points of contact are collinear with one of the centers of similitude of the circles.*

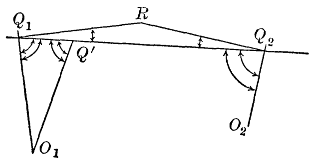

FIG. 2.

The points of contact Q_1, Q_2 and the point of intersection R of the tangents form an isosceles triangle, since $RQ_1 = RQ_2$. O_1 and O_2 being the centers of the circles, the angles $O_1Q_1Q_2$ and $O_2Q_2Q_1$ are equal, being the respective complements of two equal angles. Now, if Q' is the second point common to Q_1Q_2 and the circle O_1, we have $\angle O_1Q'Q_1 = \angle O_1Q_1Q_2 = \angle O_2Q_2Q_1$; hence O_1Q' is parallel to O_2Q_2, which proves the lemma.

This proof changes but slightly if O_1, O_2 are on opposite sides of Q_1Q_2.

17. The line $A'B'$ (Fig. 1) joining the points of contact A', B' of the two tangents OA', OB' to the two circles γ_1, γ_2, meets the line of their centers l in one of their centers of similitude (16). This is the point C_3 (12). The line AB' joining the points of contact A, B' of the two tangents OA, OB' to the two circles γ_1, γ_2 meets the line l in one of their centers of similitude (16). This is the point P_3 (13). Similarly for the points C_1, P_1, C_2, P_2. Hence:

The center of an Apollonian circle and the pole, with respect to this circle, of the Brocard diameter, are the two centers of similitude of the two other circles of Apollonius.

[1] Russell, *loc. cit.*, pp. 34, 35.

This geometry will never replace the older geometries, but with further development many new theorems may be discovered. Any one interested in modern geometry will read the work with pleasure; and detecting the few slight errors will add a little zest.

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UNIVERSITY OF PITTSBURGH.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

PROBLEMS FOR SOLUTION.

ALGEBRA.

443 Proposed by A. M. KENYON, Purdue University.

If p_r denote the sum of all the different r-factor products that can be formed from the first n natural numbers ($p_r = 0$ for $r > n$), and if

$$D_s = \begin{vmatrix} p_1 & 1 & 0 & \cdots & 0 \\ 2p_2 & p_1 & 1 & \cdots & 0 \\ 3p_3 & p_2 & p_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ sp_s & p_{s-1} & p_{s-2} & \cdots & p_1 \end{vmatrix}$$

show that

$$\sum_{i=0}^{k} (-1)^i c_i \binom{k}{i} D_{2k-i} = 0, \qquad k, n = 1, 2, 3, \cdots,$$

where $c_i = \dfrac{2k+1-i}{1+i}$ when i is even and $2n+1$ when i is odd; and $\binom{k}{i}$ is the coefficient of x^i in $(1+x)^k$.

444. Proposed by J. E. ROWE, Pennsylvania State College.

Prove that the determinant

$$\begin{vmatrix} \cot A & \cot B & \cot C \\ 1 & 1 & 1 \\ \cos^2 A & \cos^2 B & \cos^2 C \end{vmatrix} = 0,$$

where A, B, and C are the angles of a plane triangle.

GEOMETRY.

474. Proposed by LAENAS G. WELD, Pullman, Illinois.

Upon a fixed and constant base stands a system of co-planar triangles, for each of which the radius of the inscribed circle is to that of the circumscribed circle as 1 : 2. What is the locus of the vertices opposite to the given fixed base?

475. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Given two circles and a straight line, to draw a circle tangent to the line and coaxial with the two given circles.

CALCULUS.

395. Proposed by W. W. BURTON, Mercer University, Macon, Ga.

Into a full conical wine glass whose depth is a and whose angle at the base is 2α there is carefully dropped a spherical ball of such size as to cause the greatest overflow. Show that the radius of the ball is $a \sin \alpha/(\sin \alpha + \cos 2\alpha)$.

From Woods and Bailey's *A Course in Mathematics* (1907), Volume I, page 213.

396. Proposed by ELBERT H. CLARKE, Purdue University.

The length of the curve $y = x^n$ from the origin to the point (1, 1) is given by the formula

$$l = \int_0^1 \sqrt{1 + n^2x^{2n-2}}dx.$$

Our geometric intuition would tell us that the limit of this length as n becomes infinite is 2. Give a strict analytic proof that

$$\lim_{n\to\infty} \int_0^1 \sqrt{1 + n^2x^{2n-2}}dx = 2.$$

MECHANICS.

315. Proposed by H. S. UHLER, Yale University.

A solid, homogeneous, right, circular cylinder is allowed to move from rest down a circular cylindrical track which is concave upwards. Find the ratio of the radius of the track to the radius of the cylinder when the time of descent through a finite arc to the bottom is the same for the extreme cases of no slipping and zero friction. Show also that the same relation holds for a sphere descending a cylindrical or spherical surface.

316. Proposed by C. N. SCHMALL, New York, N. Y.

A body at rest at a point R begins to move towards a center of force F. The distance $RF = d$, and the force varies inversely as the distance. Two intermediate points in the path are P and Q, such that $FP = kd$, and $FQ = k^nd$. Show that the body will traverse the distance QP in a *maximum* of time if $k = 1/n^{\frac{1}{2(n-1)}}$.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

426. Proposed by HERBERT N. CARLETON, West Newbury, Mass.

Find all solutions of the equation

$$x\sqrt[x]{x} = x^x.$$

Solution by J. A. Capron, Notre Dame, Ind.

The equation may be written in the form $x^{1+(1/x)} = x^x$, or $x^{(x+1)/x} - x^x = 0$. Factoring, we have $x^x[x^{(x+1-x^2)/x} - 1] = 0$. This equation is equivalent to the two equations $x^x = 0$ and $x^{(x+1-x^2)/x} - 1 = 0$. The first of these equations is satisfied for the value of $x = -\infty$. From the second equation, we have, by taking logarithms, the equation

$$\left(\frac{x + 1 - x^2}{x}\right) \log x = 0.$$

This equation is equivalent to the three equations $1/x = 0$, $x + 1 - x^2 = 0$, and $\log x = 0$. From the first of these equations, $x = \pm \infty$; from the second, $x = (1 \pm \sqrt{5})/2$; and from the third, $x = 1$.

By substituting the values of x found above in the original equation, we see that 0 and $\pm\infty$ are to be rejected. We find, however, by inspection that $x = 1$ is a root.

Hence, the roots are 1 and $(1 \pm \sqrt{5})/2$.

Also solved by ALBERT N. NAUER, A. M. HARDING, C. E. GITHENS, V. M. SPUNAR, ELIJAH SWIFT, W. C. EELLS, G. W. HARTWELL, and the PROPOSER.

430A. Proposed by H. C. FEEMSTER, York College, Neb.

Solve the equations

$$\sum_{i=1}^{n} x_i - x_n = k + \frac{n^2 - 3n + 2}{2} d, \tag{1}$$

$$\sum_{i=1}^{n} x_i - x_{n-1} = k + \frac{n^2 - 3n + 4}{2} d, \tag{2}$$

$$\sum_{i=1}^{n} x_i - x_{n-2} = k + \frac{n^2 - 3n + 6}{2} d, \tag{3}$$

$$\cdots\cdots\cdots\cdots$$

$$\sum_{i=1}^{n} x_i - x_1 = k + \frac{n^2 - n}{2} d. \tag{n}$$

SOLUTION BY A. M. HARDING, Univ. of Arkansas.

Add the given equations and obtain

$$(n-1)\sum_{i=1}^{n} x_i = nk + \frac{n^3 - 3n^2 + n^2 + n}{2} d,$$

or

$$\sum_{i=1}^{n} x_i = \frac{n}{n-1}\cdot k + \frac{n(n-1)}{2} d.$$

Subtract each of the given equations from this equation and obtain

$$x_n = \frac{k}{n-1} + (n-1)d,$$

$$x_{n-1} = \frac{k}{n-1} + (n-2)d,$$

$$x_{n-2} = \frac{k}{n-1} + (n-3)d,$$

$$\cdots\cdots\cdots$$

$$x_2 = \frac{k}{n-1} + d,$$

$$x_1 = \frac{k}{n-1}.$$

Also solved by NATHAN ALTSHILLER, S. A. JOFFE, J. W. CLAWSON, FRANK R. MORRIS, ELBERT H. CLARKE, HORACE OLSON, N. P. PANDYA, and the PROPOSER.

431. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Form a magic square of 9 cells such that (the integers being all different) the products of the integers in the rows, columns, and diagonals shall be the same and the smallest product possible.

SOLUTION BY S. A. JOFFE, New York City.

All the nine integers will be determined if, in addition to the product p of three integers of any row (column, or diagonal), we know two extremes, say of the first row, a and b, and the central integer c.

a		b
	c	

The extremes of the third row will then be p/ac and p/bc; of the second row will be bc/a and ac/b; and those of the second column will be p/ab and ab/c.

Hence the product of the integers in the second row equals

$$\frac{bc}{a} \cdot c \cdot \frac{ac}{b} = c^3,$$

and therefore

$$p = c^3,$$

so that the magic square becomes

a	$\frac{c^3}{ab}$	b
$\frac{bc}{a}$	c	$\frac{ac}{b}$
$\frac{c^2}{b}$	$\frac{ab}{c}$	$\frac{c^2}{a}$

If we disregard the central integer c, the remaining eight integers must therefore form four pairs: (1) m_1 and c^2/m_1; (2) m_2 and c^2/m_2; (3) m_3 and c^2/m_3; (4) m_4 and c^2/m_4. Taking now for m_1, m_2, m_3 and m_4 the smallest four integers, *i. e., 1, 2, 3 and 4,* we find that the *smallest square* number c^2, divisible by these four integers, is *36*, and consequently $c = \sqrt{36} = 6$.

However, no corner number, say a, can be unity; because, if $a = 1$ then ac/b and ab/c become c/b and b/c, which must be simultaneously integers, and this is impossible unless $b = c$; but the latter equality is excluded by the condition of the problem. Taking $a = 2$, $b = 3$ and $c = 6$, we find the required magic square with the smallest product possible to be

2	36	3
9	6	4
12	1	18

Also solved by HERBERT N. CARLETON, N. P. PANDYA.

432. Proposed by C. N. SCHMALL, New York City.

There are n straight lines in a plane, no two of which are parallel and no three of which are concurrent. Their points of intersection being joined show that the number of new lines drawn is $\frac{1}{8}n(n-1)(n-2)(n-3)$.

Solution by Laenas G. Weld, Pullman, Illinois.

Each of the n given lines intersects each of the other $(n-1)$ lines, determining (since each point is thus twice determined) points to the number of $\dfrac{n(n-1)}{2}$.

Let P_{ik} be the point determined by the lines l_i and l_k. From P_{ik} there may be drawn $R\left(=\dfrac{n(n-1)}{2}-1\right)$ lines to the other points. But the line l_i contains $(n-2)$ of these other points and the line l_k the same number. Hence the given lines l_i and l_k account for $2(n-2)$ of the above R lines and the number of new lines is

$$R-2(n-2)=\left(\frac{n(n-1)}{2}-1\right)-2(n-2).$$

Since there are $\dfrac{n(n-1)}{2}$ points such as P_{ik} the total number of new lines (since each is thus counted twice) is

$$\frac{1}{2}\cdot\frac{n(n-1)}{2}\left\{\left(\frac{n(n-1)}{2}-1\right)-2(n-2)\right\},$$

or

$$\frac{1}{8}n(n-1)(n-2)(n-3).$$

Also solved by J. W. Clawson, Herbert N. Carleton, H. C. Feemster, Walter C. Eells, Paul Capron, Frank R. Morris, Elbert H. Clarke, N. P. Pandya, and Horace Olson.

GEOMETRY.

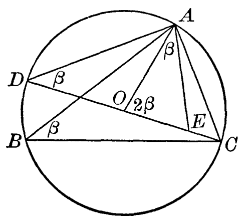
461. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Prove by means of any inscribed triangle the following trigonometrical relations:

$\sin 2\beta = 2\sin\beta\cos\beta$; $\cos 2\beta = \cos^2\beta - \sin^2\beta$; $\sin 3\beta = 3\sin\beta - 4\sin^3\beta$; $\cos 3\beta = 4\cos^3\beta - 3\cos\beta$.

Solution by J. W. Clawson, Collegeville, Pa.

Let ABC be the triangle. Join A and C to O, the center of the circumcircle. Extend CO to meet the circle at D. Draw AD. Draw AE making angle OAE equal to angle ABC. Denote angle ABC by β. Call the radius R.



(1) $$\sin 2\beta = \sin AOC = \sin ACD \cdot \frac{AC}{R} = \frac{AC \cdot AD}{2R^2} = 2\frac{AC}{2R} \cdot \frac{AD}{2R} = 2 \sin\beta \cos\beta.$$

(2) $$\cos 2\beta = \cos AOC = \frac{R^2 + R^2 - \overline{AC}^2}{2R \cdot R} = \frac{4R^2 - \overline{AC}^2 - \overline{AC}^2}{4R^2} = \frac{\overline{AD}^2 - \overline{AC}^2}{4R^2}$$
$$= \left(\frac{AD}{2R}\right)^2 - \left(\frac{AC}{2R}\right)^2 = \cos^2\beta - \sin^2\beta.$$

(3) $$\sin 3\beta \sin AEC = \sin AEO = \sin\beta \cdot \frac{R}{OE}.$$

Now

$$\frac{OE}{R} = \frac{AE}{AD} = \frac{AE}{2R\cos\beta}; \quad i.\ e.,\ AE = 2\overline{OE}\cos\beta.$$

Again

$$OE^2 = \overline{AE}^2 + R^2 - 2\overline{AE} \cdot R\cos\beta = 4\overline{OE}^2\cos^2\beta + R^2 - 4\overline{OE} \cdot R \cdot \cos^2\beta;$$

$$\therefore\ OE = \frac{R}{4\cos^2\beta - 1};$$

$$\therefore\ \sin 3\beta = \sin\beta(4\cos^2\beta - 1) = 3\sin\beta - 4\sin^3\beta.$$

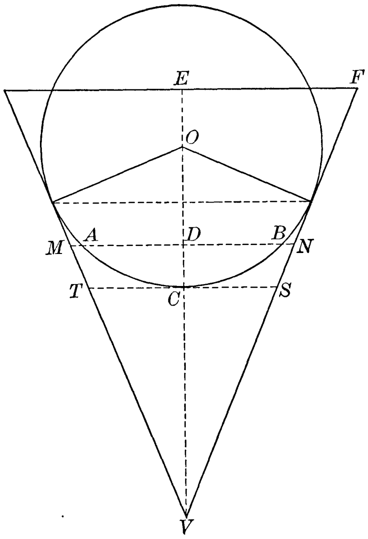
(4) $$\cos 3\beta = \frac{R^2 - \overline{AE}^2 - \overline{OE}^2}{2\overline{OE} \cdot AE} = \frac{(4\cos^2\beta - 1)^2 - 1 - 4\cos^2\beta}{4\cos\beta} = 4\cos^3\beta - 3\cos\beta.$$

462. Proposed by DANIEL KRETH, Wellman, Iowa.

A conical glass, the diameter of the base of which is 5 inches and altitude 6 inches, is one-fifth full of water. If a sphere 4 inches in diameter is dropped into it, how much of the vertical axis of the glass is immersed?

SOLUTION BY J. A. CAPRON, Notre Dame University.

Let MN be the level of the water after the sphere is dropped; $R = 5/2$ inches = radius of base of given cone; $r = 2$ inches = radius of sphere; r_1 = radius of base of cone VMN; x = height of spherical segment ACB; α = semivertical angle of cone; and $CV = a$.



Then volume of cone $VMN = V_c = \frac{1}{3}\pi r_1^2(a+x)$; volume of segment $ACB = V_s = \pi x^2(r - x/3)$; and volume of given cone $= V = \frac{1}{3}\pi R^2 h$, where $h = 6$ inches.

Hence $V_c - V_s = \frac{1}{5}V$, since $V_c - V_s$ = volume of water. Substituting, we get

$$r_1^2(a + x) - x^2(3r - x) = \frac{R^2h}{5}.$$

Since

$$VC = VO - CO \quad \text{and} \quad VO = \frac{r}{\sin\alpha},$$

we get

$$a = r\left(\frac{1}{\sin\alpha} - 1\right),$$

but since

$$\tan\alpha = \frac{5}{12}, \quad \sin\alpha = \frac{5}{13},$$

therefore

$$a = 2\left(\frac{13}{5} - 1\right) = \frac{16}{5}.$$

From triangles VEF and VDN, we get

$$r_1 = \frac{(a + x)}{h}R, \quad \text{or} \quad r_1 = \frac{\left(\frac{16}{5} + x\right)\frac{5}{2}}{6} = \frac{4}{3} + \frac{5x}{12}.$$

Substituting these values in the above equation and reducing, we get

$$845x^3 - 3120x^2 + 3840x - 1304 = 0.$$

To solve this equation, let

$$x = y + \frac{108}{169}, \quad \text{or} \quad x = y + \frac{16}{13}.$$

Substituting and reducing, we get

$$169y^3 = -\frac{16^3}{13} + 260.8, \quad \text{or} \quad 13^3y^3 = -705.6.$$

Hence,

$$y = -\frac{\sqrt[3]{705.6}}{13} = -\frac{2}{13}\sqrt[3]{88.2}.$$

Then

$$x = y + \frac{16}{13} = \frac{16 - 2\sqrt[3]{88.2}}{13} = .54595.$$

Hence, the required height $= a + x = 3.2 + .54595 = 3.74595$ inches.

Excellent solutions were received from NATHAN ALTSHILLER, C. N. SCHMALL, HERBERT N. CARLETON, J. W. CLAWSON, HORACE OLSON, and PAUL CAPRON.

CALCULUS.

375. Proposed by V. M. SPUNAR, Chicago, Illinois.

Solve the differential equation

$$x^2(a - bx)\frac{d^2y}{ax^2} - 2x(2a - bx)\frac{dy}{dx} + 2(3a - bx)y = 6a^2.$$

I. SOLUTION BY H. T. BIGELOW, La Fayette, Indiana.

Let

$$y = \sum_{n=0}^{\infty} c_n x^n, \quad \frac{dy}{dx} = \sum_{n=0}^{\infty} nc_n x^n, \quad \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}.$$

Substituting and collecting terms we have,

$$\sum_{n=0}^{\infty} a(n^2 - 5n + 6)c_n x^n \equiv \sum_{n=0}^{\infty} b(n^2 - 3n + 2)c_n x^{n+1} + 6a^2.$$

Equating the coefficients of each power of x on the two sides of this equality, we have,

$$\begin{array}{ll} \text{for } x^0 & 6ac_0 = 6a^2; \\ \text{for } x & 2ac_1 = 2bc_0; \\ \text{for } x^2 & 0ac_2 = 0bc_1; \\ \text{for } x^3 & 0ac_3 = 0bc_2; \end{array}$$

and

$$\text{for } x^i,\ (i^2 - 5i + 6)aci = (i^2 - 5i + 6)bc_{i-1}.$$

Solving these equations;

$$c_0 = a, \qquad c_1 = b, \qquad c_2 \text{ is arbitrary}, \qquad c_3 \text{ is arbitrary},$$

$$c_4 = \frac{b}{a}c_3, \qquad c_5 = \frac{b^2}{a^2}c_3, \qquad c_6 = \frac{b^3}{a^3}c_3, \quad \text{etc.}$$

Hence,

$$y = a + bx + c_2x^2 + c_3\left(x^3 + \frac{b}{a}x^4 + \frac{b^2}{a^2}x^5 + \cdots\right)$$

$$= a + bx + c_2x^2 + \frac{c_3x^3}{1 - \dfrac{b}{a}x}, \qquad \text{if } |x| < \frac{a}{b}.$$

Renaming the constants,

$$y = a + bx + Ax^2 + \frac{Bx^3}{a - bx},$$

where A and B are arbitrary. By substitution, it is readily verified that this actually is the solution for all values of x, except of course $x = \dfrac{a}{b}$.

II. Solution by W. W. Beman, Ann Arbor, Mich.

This problem appears in Article 69 of Forsyth's *Differential Equations* and may be solved by the method there indicated. The resolution into factors may be effected more easily by symbolic methods.

Putting

$$x\frac{d}{dx} \equiv \theta,$$

the equation becomes

$$[a(\theta^2 - 5\theta + 6) - bx(\theta^2 - 3\theta + 2)]y = 6a^2,$$

which may be written in three different ways:

$$[a(\theta - 3) - bx(\theta - 1)](\theta - 2)y = 6a^2, \text{ or}$$

$$(\theta - 3)[a(\theta - 2) - bx(\theta - 1)]y = 6a^2, \text{ or}$$

$$(\theta - 2)[a(\theta - 3) - bx(\theta - 2)]y = 6a^2.$$

Hence, three particular integrals of the equation, with second member 0, are

$$y_1 = x^2, \qquad y_2 = \frac{x^2}{a - bx}, \qquad y_3 = \frac{x^3}{a - bx},$$

only two of which are independent.

We may now put $y = y_1 v$, or $y = y_2 w$, or $y = y_3 z$, etc.: or $(\theta - 2)y = v$, or[1] $[a(\theta - 2) - bx(\theta - 1)]y = w$, or $[a(\theta - 3) - bx(\theta - 2)]y = z$, etc.

The substitution $y = y_2 w$ leads to the *normal* form, which could have been obtained in the ordinary way,

$$\frac{d^2w}{dx^2} = \frac{6a^2}{x^4}.$$

Hence,

$$w = A + Bx + \frac{a^2}{x^2}.$$

$$\therefore \quad y = \frac{x^2}{a - bx}\left[A + Bx + \frac{a^2}{x^2}\right].$$

Again, the equation may be written

$$[(\theta - 2)(\theta - 3)(a - bx)]y = 6a^2.$$

$$\therefore \quad (a - bx)y = Ax^2 + Bx^3 + a^2.$$

Also solved by V. M. Spunar, Elijah Swift, A. M. Harding, and Elmer Schuyler.

376. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that

$$\frac{1}{z} - \frac{1}{z}(1 - 2xz + z^2)^{1/2} = x + \frac{z}{2}\left(\frac{x^2 - 1}{1 - xz}\right) + \sum_{n=2}^{n=\infty} \frac{1\cdot 3\cdot 5\cdot \cdots \cdot(2n - 3)}{2\cdot 4\cdot 6\cdot \cdots \cdot 2n}(x^2 - 1)^n\left(\frac{z}{1 - xz}\right)^{2n-1}$$

Solution by Elijah Swift, University of Vermont.

First of all the coefficients of the terms of the infinite series are the same as those in the expansion of $-(1 - a)^{\frac{1}{2}}$. These lead us to expand

$$\left\{1 - (x^2 - 1)\left(\frac{z}{1 - xz}\right)^2\right\}^{1/2}$$

by the Binominal theorem, which gives

$$1 - \tfrac{1}{2}(x^2 - 1)\left(\frac{z}{1 - xz}\right)^2 - \left\{\frac{1\cdot 1}{2\cdot 4}(x^2 - 1)^2\left(\frac{z}{1 - xz}\right)^4 + \cdots\right\}.$$

Noting that the terms in the bracket are the corresponding terms in the infinite series each multiplied by $z/(1 - xz)$, we find that

$$\sum_{n=2}^{\infty} \frac{1\cdot 3\cdot 5\cdot \cdots \cdot(2n - 3)}{2\cdot 4\cdot 6\cdot \cdots \cdot 2n}(x^2 - 1^n)\left(\frac{z}{1 - xz}\right)^{2n-1}$$
$$= -\frac{1 - xz}{z}\left\{1 - (x^2 - 1)\left(\frac{z}{1 - xz}\right)^2\right\}^{1/2} + \frac{1 - xz}{z} - \tfrac{1}{2}(x^2 - 1)\left(\frac{z}{1 - xz}\right).$$

[1] These two forms of solution are indicated in the German Edition of Forsyth.

Substituting this value, the result follows by a simple algebraic reduction. The result is true only when the series converges, *i. e.*, when $|(x^2-1)(z/(1-xz))^2|<1$ or $x<(1+z^2)/2z$.

Also solved by S. A. JOFFE, GEO. W. HARTWELL, ALBERT N. NAUER, and A. M. HARDING.

Note.—This problem should have been placed in the Algebra department. EDITORS.

377. Proposed by W. D. CAIRNS, Oberlin College.

It is required to find a curve of the form $y=x(x-a)(x-b)$ such that the abscissas of the maximum and minimum values, as well as a and b, shall be positive integers.

SOLUTION BY RALPH D. BEETLE, Dartmouth College.

The abscissas of the minimum and maximum values are

$$(1)\qquad x_1=\tfrac{1}{3}[a+b+\sqrt{a^2-ab+b^2}],\qquad x_2=\tfrac{1}{3}[a+b-\sqrt{a^2-ab+b^2}].$$

If a and b are positive integers, so also are x_1 and x_2 if, and only if, $a+b$ is divisible by 3 and a^2-ab+b^2 is a perfect square divisible by 9. These conditions are equivalent to the simpler ones that a and b be divisible by 3 and a^2-ab+b^2 be a perfect square. If we write

$$(2)\qquad a=3h,\ b=3k$$

our problem is reduced to the finding of positive integral values of h and k such that h^2-hk+k^2 is a perfect square. We note at once that, if a and b are equal, it is sufficient that they be multiples of 3. We may find all solutions of the problem as follows:

Without loss of generality, we may assume that $h\leqq k$, so that $h^2-hk+k^2\leqq k^2$. If we put $h^2-hk+k^2=(k-r)^2$, it is found that

$$(3)\qquad h=\tfrac{1}{2}[k\pm\sqrt{k^2-8rk+4r^2}].$$

In order that h be an integer, we must have $k^2-8rk+4r^2=t^2$, where t is an integer, or $(k-4r)^2-t^2=12r^2$. It is evident that k and t are both even or both odd, and we may therefore put

$$(4)\qquad k-4r=n+s,\qquad t=n-s,$$

where n and s are positive integers. Then

$$(5)\qquad ns=3r^2,$$

and the values of a, b, x_1, x_2 in terms of r, n, s are found from (1), (2), (3), (4) to be either

$$a=3n+6r\qquad b=3n+12r+3s,\qquad x_1=3n+9r+2s,\qquad x_2=n+3r,$$

or

$$a=6r+3s,\qquad b=3n+12r+3s,\qquad x_1=2n+9r+3s,\qquad x_2=3r+s.$$

Since the two sets differ only in the interchange of n and s, only one set is needed in view of the symmetric roles of n and s in the relation (5). We may therefore state our result as follows:

If n, s and r are integers such that $n > 0$, $s \geqq 0$, $r \geqq 0$, $ns = 3r^2$, and if $a = 3n + 6r$, $b = 3n + 12r + 3s$, the abscissas of the minimum and maximum points of the curve, $y = x(x - a)(x - b)$, are the positive integers $x_1 = 3n + 9r + 2s$ and $x_2 = n + 3r$.

The smallest unequal values of a and b are 9 and 24. In the following table are the solutions which correspond to $r = 1$ and $r = 2$.

n	1	3	1	2	3	4	6	12
s	3	1	12	6	4	3	2	1
r	1	1	2	2	2	2	2	2
a	9	15	15	18	21	24	30	48
b	24	24	63	48	45	45	48	63
x_1	18	20	45	36	35	36	40	56
x_2	4	6	7	8	9	10	12	18

The consideration of negative values of n and s is unnecessary, since it can be proved that no additional solutions of the problem are thus obtained.

Also solved by ALBERT N. NAUER, W. C. EELLS, JOSEPH B. REYNOLDS, ELMER SCHUYLER ELIJAH SWIFT, HORACE OLSON, and the PROPOSER.

MECHANICS.

292. Proposed by C. N. SCHMALL, New York City.

In a bombardment, a battleship directs its fire at a fort standing on a hill whose height is a feet above the sea level. The angle of elevation of the fort is found to be ϕ. If the initial velocity of the projectile is v, show that the fort will *not* be struck if $v < \sqrt{ag(1 + \csc \phi)}$.

SOLUTION BY H. S. UHLER, Yale University.

Let the coördinate plane contain the fort and the vertical through the ship. Also let θ denote the angle which the rifle makes with the positive (upward) direction of the axis of y. At any point of the trajectory

$$x = v \sin \theta \cdot t,$$

$$y = v \cos \theta \cdot t - \tfrac{1}{2}gt^2,$$

hence

$$gx^2 - v^2 \sin 2\theta \cdot x + 2v^2 \sin^2 \theta \cdot y = 0.$$

This parabola will pass through the fort ($x = a \cot \phi$, $y = a$) when

$$a^2 g \cot^2 \phi \cdot \cot^2 \theta - 2av^2 \cot \phi \cdot \cot \theta + a(2v^2 + ag \cot^2 \phi) = 0.$$

The roots of this quadratic in $\cot \theta$ will be complex if

$$(v^2 - ag)^2 - (ag \csc \phi)^2$$

is negative. Since this expression may be written

$$[v^2 + ag(\csc \phi - 1)] \cdot [v^2 - ag(\csc \phi + 1)],$$

it is seen at once that the fort will not be hit when

$$v < \sqrt{ag(1 + \csc \phi)}.$$

295. Proposed by B. F. FINKEL, Drury College.

A homogeneous hollow cylinder whose inner radius is half its outer radius, rolls without slipping down a plane inclined at an angle α with the horizontal. Find its acceleration.

SOLUTION BY JOS. B. REYNOLDS, Lehigh University.

If in a section perpendicular to the axis of the cylinder, O is the center and A the point of contact with the plane, we shall have for the moment of inertia about O,

$$I_O = \frac{\pi}{2} e \left(r^4 - \frac{r^4}{2^4} \right),$$

and for the mass

$$m = \frac{W}{g} = \pi e \left(r^2 - \frac{r^2}{2^2} \right),$$

whence

$$I_O = \frac{5}{8} \frac{W}{g} r^2$$

and therefore

$$I_A = I_O + \frac{W}{g} r^2 = \frac{13}{8} \frac{W}{g} r^2.$$

Now taking moments about A, we have $Wr \sin \alpha = I_A \ddot{\theta}$ where $\ddot{\theta}$ is the angular acceleration, or

$$Wr \sin \alpha = \frac{13}{8} \frac{W}{g} r^2 \ddot{\theta},$$

whence $r\ddot{\theta} = \frac{8}{13} g \sin \alpha$, the linear acceleration of the center of the cylinder.

Also solved by H. S. UHLER.

QUESTIONS AND DISCUSSIONS.

EDITED BY U. G. MITCHELL, University of Kansas.

REPLIES.

24. The following facts are significant:

(1) The New England Association of Mathematics Teachers has appointed a committee "to investigate the current criticisms of high school mathematics."

(2) A committee of the Council of the American Mathematical Society has under consideration the question "whether any action is desirable on the part of the Society in the matter of the movement against mathematics in the schools."

(3) At the recent meeting in Cincinnati of the National Education Association an iconoclastic discussion on the topic: "Can algebra and geometry be reorganized so as to justify their retention for high school pupils not likely to enter technical schools?" aroused approbation and applause. An outline of the remarks by one of the speakers was printed in a previous issue.

In view of these facts what should be done by those who believe in the value of mathematics as a general high school study?

REPLY BY HARRISON E. WEBB, Central High School, Newark, N. J.

High-school teachers of mathematics should (1) do their utmost to restore the older ideal of high-school education as a serious business for the individual student; (2) relate their science as closely as possible to other studies which are pursued simultaneously; (3) preserve as closely as possible the essential continuity of the entire high-school mathematical course; (4) if necessary, as it may be, to accomplish these ends, eliminate excessive complications and omit less important topics; (5) give over entirely the notion that their subjects are absolutely logical, but make them positive so far as they go; and (6) believe in mathematics as possessing an incomparable moral value.

DISCUSSIONS.

RELATING TO INTERPOLATION IN TABLES OF TWO ARGUMENTS.

BY IRWIN ROMAN, Chicago, Ill.

Anyone who works with tables of two arguments has probably found interpolation inconvenient. The following method renders it possible with almost the same facility as interpolation in one argument. Let the function be given for four adjacent values, two in each argument. This may be represented schematically as follows. Let $A = f(x_1, y_1)$, $B = f(x_2, y_1)$, $C = f(x_1, y_2)$, $D = f(x_2, y_2)$ and let it be required to find $G = f(x_1 + \Delta x, y_1 + \Delta y)$ where

$$\frac{\Delta x}{x_2 - x_1} = e_x < 1 \qquad \text{and} \qquad \frac{\Delta y}{y_2 - y_1} = e_y < 1.$$

For convenience, let

$$x_1 \leqq x_1 + \Delta x \leqq x_2 \quad \text{and} \quad y_1 \leqq y_1 + \Delta y \leqq y_2.$$

Let

$$E = f(x_1, y_1 + \Delta y) \quad \text{and} \quad F = f(x_2, y_1 + \Delta y).$$

The usual way of finding G is to interpolate E from A and C, to interpolate F from B and D and finally, to interpolate G from E and F. This gives

$$E = A + e_y(C - A),$$

$$F = B + e_y(D - B),$$

$$G = E + e_x(F - E).$$

Carrying this step farther gives

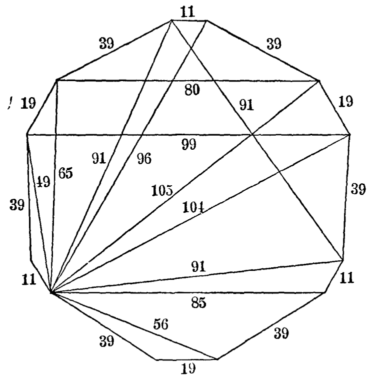
$$\begin{aligned} G &= A + e_y(C - A) + e_x[B + e_y(D - B) - A - e_y(C - A)] \\ &= A + e_y(C - A) + e_x(B - A) + e_xe_y[(A + D) - (B + C)]. \end{aligned}$$

The last term is usually negligible, especially if $f(x, y)$ changes in the same sense for both x and y. This is simple to remember as it involves an interpolation horizontally and one vertically. It avoids the necessity of calculating E and F.

RELATING TO A GEOMETRIC REPRESENTATION OF INTEGRAL SOLUTIONS OF CERTAIN QUADRATIC EQUATIONS.

By Norman Anning, Chilliwack, British Columbia.

The cyclic dodecagon shown in the accompanying figure is an interesting special solution of the problem: To locate n points in the plane so that the $\binom{n}{2}$ distances shall be integral.



The fact that Ptolemy's Theorem may be verified in no less than 81 different ways entitles the group of numbers to rank with the most dignified of magic squares. The 13 numbers are all found among the integral solutions of

$$x^2 \pm xy + y^2 = 7^2 \cdot 13^2.$$

In like manner 40 integers which occur among the solutions of

$$x^2 \pm xy + y^2 = 7^2 \cdot 13^2 \cdot 19^2$$

may be exhibited as the sides and diagonals of a cyclic 24-gon. The sides in order are: 96, 361, 299, 209, 249, 209, 299, 361, 96, 361, ...

That a study of

$$x^2 + xy + y^2 = 7^2 \cdot 13^2 \cdot 19^2 \cdot 31^2$$

would yield a similar 48-gon is probable.

NOTES AND NEWS.

EDITED BY W. D. CAIRNS, Oberlin, Ohio.

Dr. W. S. FRANKLIN has resigned from his position as professor of physics in Lehigh University.

A "School algebra—first course," by RIETZ, CRATHORNE and TAYLOR appears in the press of Henry Holt and Company.

Dr. G. H. GRAVES, of Columbia University, has been appointed instructor in mathematics in Purdue University.

President ROBERT S. WOODWARD, of Carnegie Institution, and Professor ARTHUR G. WEBSTER, of Clark University, have been chosen from the American Mathematical Society to membership on the Naval Advisory Board of Inventions.

JOHN HOWARD VAN AMRINGE, professor emeritus of mathematics in Columbia University, a member of the Columbia faculty for fifty years up to his retirement in 1910, died September 10.

Mr. F. E. CARR has returned to a position in the department of mathematics in Oberlin College after two years of study in the graduate school of the University of Chicago.

In *Science* for September 10 Professor G. A. MILLER discusses critically the article by Professor E. Borel in the April number of *Revue du Mois*, in which he asserts that in this country, as in Germany, the prevailing tendency in estimating scientific achievements is to emphasize quantity rather than quality.

In the May–July number of *Rendiconti del Circolo Matematico di Palermo* appear two articles by Americans: "The restricted problem of three bodies," by Professor G. D. BIRKHOFF, and "On the degree of approximation to discontinuous functions by trigonometric sums," by CHARLES E. WILDER, of Cambridge, Mass.

The Macmillan Company has published "Modern instruments and methods of calculation," a hand-book of the Napier tercentenary exhibition, under the editorship of E. M. Horsburgh. The same press announces an "Historical introduction to mathematical literature" by Professor G. A. MILLER.

An address entitled "Forty years' fluctuations in mathematical research," by Professor H. S. WHITE, which was read before the Vassar faculty club and the Columbia University mathematical colloquium, is printed in *Science* for July 23.

N. C. GRIMES, formerly professor of mathematics in the University of Arizona and lately assistant in mathematics in the University of Illinois, has accepted a

position in the University of Oregon as professor of mathematics and secretary to the President.

Dr. NATHAN ALTSHILLER, of the mathematical staff in the University of Washington, has resigned there to accept an instructorship in mathematics in the University of Colorado, taking effect in September of the present year.

A meeting of Missouri teachers of collegiate mathematics was also held at St. Louis on Saturday, November 27, in conjunction with the Southwestern Section of the American Mathematical Society. A further report of this meeting will appear in the December MONTHLY.

Professor F. CAJORI has returned to his teaching in Colorado College after a year abroad. After attending the Napier tercentenary at Edinburgh, he spent most of the year in Oxford and London in research connected with the history of mathematics. Other parts of the year he spent in travel in France, Italy, Switzerland, and Germany.

The Fall Announcement of New Macmillan Books includes the title "Historical Introduction to Mathematical Literature," by Professor G. A. Miller. The work "aims to give a brief account of the most important modern mathematical activities such as the mathematical societies, mathematical congresses, and periodical publications."

J. K. SINCLAIR, professor emeritus of mathematics in Worcester Polytechnic Institute, died September 12 after two years of increasingly failing health. Because of his sickness the leave of absence of his daughter Dr. M. E. Sinclair of Oberlin College was extended for a year, her place being supplied by Mr. L. M. Coffin, a graduate of the University of Maine and a master of arts in the University of Michigan.

The North-Eastern Ohio Teachers' Association was held in Cleveland October 22 and 23. At the departmental meeting for mathematics Mr. P. C. Bickel of the Alliance high school presented a paper on "How much time per week for supervision in the study of mathematics?" and Mr. H. A. Tuttle of Rayen high school, Youngstown, a paper on "The present trend in mathematical teaching—is it wise?"

In *School Science and Mathematics* for October J. H. Weaver, of the West Chester (Pa.) high school, presents the principal methods of the Greeks for attacking "The trisection problem"; M. O. Tripp, of Olivet College, points out the importance of "Symmetry in elementary geometry"; and J. V. Collins, of the Stevens Point (Wis.) State Normal School, discusses "Rational *vs.* mechanical methods in teaching mathematics."

In the *Proceedings* of the Society for the Promotion of Engineering Education appears the preliminary report of the committee on coöperation with secondary schools, under the chairmanship of H. E. WEBB, Central Commercial and Manual Training High School, Newark, N. J. The report urges the fuller recognition of the worth of high-school training for its own sake, a greater willingness to accept at its face value the judgment of a high-school teacher as to the ability of his pupils, and the elimination of the waste between the lower and higher institutions which is at present involved in the repetition of numerous topics in mechanical drawing, manual training, mathematics, physics and chemistry.

The sixth number of the first volume of the *Proceedings of the National Academy of Sciences* contains an article by Professor F. R. MOULTON on the "Solution of an infinite system of differential equations of the analytic type" and another by Professor HENRY BLUMBERG entitled "On the factorization of various types of expressions." The seventh number contains articles by Professor W. B. FORD on "The representation of arbitrary functions by definite integrals" and by Professor W. D. MACMILLAN on "Some theorems connected with irrational numbers." The eighth number contains an article by Professor H. S. WHITE on "Seven points on a twisted cubic curve."

The first meeting of the Kansas association of teachers of college mathematics was held at Topeka, Kansas, November 12. This meeting was the result of a movement initiated in the spring of 1915 for the improvement of teaching collegiate mathematics in the colleges of Kansas. It is a part of a nation wide movement having the same end. The temporary committee of organization consisted of Professor A. J. HOARE of Fairmount College, Professor T. J. MERGENDAHL of Emporia College, and Professor S. LEFSCHETZ of the University of Kansas. About fifteen representatives attended the meeting and a permanent organization was effected with Professor HOARE as president. A paper was presented by Professor W. A. HARSHBARGER of Washburn College on "What courses for the mathematical student beyond the calculus." The paper was discussed by Professor B. L. REMICK of the State Agricultural College and Professor W. H. GARRET of Baker University. It will appear in the December issue of the MONTHLY as a reply to question 27 in the department of Questions and Discussions.

This action of the college teachers of mathematics in Kansas is the first step in a movement that promises to grow rapidly. Definite plans are already formed for a similar organization in Ohio during the Christmas holidays, in conjunction with the meeting to be called for organizing a new national mathematical association, which is to be held at Columbus on Thursday, December 30, at ten o'clock in Page Hall of Ohio State University.

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ERRATA IN VOLUME XXII.

Page 61, line 4 down, "$c = 1$" should read $c = 2$.

Page 68, line 13 down, "$f''(x) = -c(2n-1)\pi$" should read $f''(x) = e^{-c(2n-1)}\pi$.

Page 130, line 6 down, problem "430" should read 430A.

Page 131, line 4 down, "$\pi^3/12^3$" should read $\pi^3/12$.

Page 137, line 6 down, "$3a^2 + 2ab = A$" should read $3a^2 = 2ab = A$. Line 14 down, "dy/ax" should read dy/dx.

Page 161, line 14 down, problem "463" should read 463A.

Page 167, in equation (5), "DF/EF" should read DE/EF. Line 6 up, the first radical sign should extend over the numerator only.

Page 205, line 17 up, "log 12/log 2" should read log 12/log 6. Line 12 up, "Irvin" should read Irwin.

Page 251, line 3 up, the first word should be "or."

Page 292, line 18 down, "Cantorion" should read Cantorian.

Page 293, line 17 up, "creatrive" should read creative.

Page 294, line 1 down, "dissusions" should read discussions.

Page 319, line 11 down, "$\frac{108}{169}$" should read $\frac{208}{169}$.

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AN EXPERIMENT IN CORRELATING FRESHMAN MATHEMATICS.

By F. L. GRIFFIN, Reed College.

Perhaps everyone would agree that it is desirable (if it is feasible) to introduce the study of calculus much earlier than is customary, to correlate it more closely with the elementary subjects and these in turn more closely with each other, and to exhibit actual uses for each topic at the time it is taught.

Quite commonly. unless a student presents trigonometry and college algebra for admission to college, he does not reach integral calculus until the middle of his sophomore year, and in some excellent colleges not until his junior year. Consequently not even the simplest integrations can be employed in the early physics courses. Moreover, students of the natural or social sciences who need a little knowledge of integral calculus must take two or three years of college mathematics in order to get it. Those who can devote but one year to mathematics in college must usually stop with no idea at all of mathematics beyond trigonometry and college algebra, and a very inadequate conception of the uses of even these subjects.

In fact, how *can* the matter of applications be adequately treated when the different branches are studied separately? Few practical problems depend for their solution upon a single branch of mathematics. Such topics as trigonometric analysis show their full value only by the closest sort of correlation.

Again, when we see 200 freshmen spend a couple of weeks decomposing the most complicated fractions into partial fractions, in order that 20 of their number may be able to use this process two years later in the integral calculus, we not only question the value of the process for the freshmen but we are inclined to doubt the law of the conservation of energy. This is an extreme case; but, wherever a student must go through college algebra, trigonometry and analytic geometry before getting any calculus, similar difficulties exist.

Of course, several institutions have partially overcome these difficulties for students of the sciences and others who wish to get a glimpse of higher mathematics, by offering survey courses designed primarily for those who do not expect to specialize in mathematics. But how about the students who *do* intend to specialize—who are expecting to work in engineering, astronomy, or physics,

or to teach mathematics? Are not these the very ones who should begin using the fine tools of the calculus as soon as possible in their other studies—the very ones, moreover, who need to acquire the sort of familiarity with the calculus that comes only from using it persistently during several years? Again, are not these specialist students the ones to whom we should be most anxious to show the relations and meaning of the various branches of mathematics all along the way; that is, the ones for whom we should correlate the subjects most carefully?

Although there is some difference of opinion upon this point, it seems to the writer that a combination course which gives a preliminary bird's-eye view of the field, will, if worked out with sufficient care, prove to be not only the best final course for non-specialist students but also the best introductory course for specialist students. To be effective such a course must not be a mere collection of parts of the several subjects; it must correlate the topics and have a reasonable degree of unity and coherence. And, to meet the needs mentioned above, it must neither presuppose trigonometry nor omit integral calculus from the work of the first year.

Possibly any feeling which may exist that the old separate courses give a better foundation for specializing in mathematics is due to the difficulty of arranging a general course so as to make adequate provision for thorough drill on the various topics. But this difficulty can be overcome. Again, it is often doubted that much calculus can be taught before college algebra, trigonometry and analytics. To be sure, we cannot teach all of the ordinary calculus, but enough of it can be taught to be very serviceable and illuminating. And the rest can be given in connection with trigonometry, algebra, and analytic geometry.

An outline of a course which has been evolved and taught for several years at Reed College will serve to show one way in which this sort of thing can be done effectively. The subjects treated are: (1) Some practical uses of graphs; (2) Some important limit concepts; (3) Differentiation; (4) Integration; (5) Trigonometric functions of acute angles; (6) Logarithms; (7) Further differentiation and integration; (8) Uses of rectangular coördinates; (9) Solution of equations; (10) Polar coördinates and trigonometric functions in general; (11) Trigonometric analysis; (12) Definite integrals; (13) Progressions and series; (14) Probability and least squares.

We start with graphs because of their simplicity, obvious utility, and natural association with the foundations of the calculus. To illustrate: the student draws a velocity-time graph from a table, and uses it for interpolating, getting average and instantaneous accelerations, and finding the total distance traveled. He is obliged to consider the questions: What is an instantaneous rate? What is the true average velocity during any interval, if the instantaneous velocity increases as in the graph? These questions are further emphasized by attempting to calculate approximately the instantaneous rate of an algebraic function and the area under its graph.

Thus he is led inevitably to the careful analysis of several familiar ideas, such as speed, rate, tangent line, mean value, length, area, and volume. And the

formulation of these concepts in terms of limits shows him both their real meaning (the understanding of which, by the way, is a thing worth while for every educated person), and also how he must proceed in order to calculate them. When he has seen these he has the basic ideas of the calculus.

In fact, before he suspects that he is dealing with anything so terrible, the student has differentiated numerous simple functions (polynomials in x and $1/x$) by the increment method, and has solved easy problems on instantaneous rates, differential corrections and maxima and minima. Of course these have to be selected with great care. After getting the formulas for differentiating positive and negative powers of x at sight he meets simple problems on motion, flexure of beams, and other rates, in which he needs to differentiate two or three times in succession; and other problems in which he needs to differentiate a power of a function.

The need of anti-differentiation soon arises in problems of motion, and also in calculating areas, work, volumes, fluid pressure, etc. No mention of definite integrals is made here; in fact, the practice of determining a constant of integration each time it arises both calls attention to its importance and helps the student later on to see the meaning of the term $-\phi(a)$ in $\int_a^b \phi'(x)dx$.

The student now meets some problems which he cannot solve completely because he does not know the anti-derivative, and some in which he cannot yet express the functional relation between the sides and angles of a triangle. He is reminded that he plotted numerous functions by using tabulated values, for which he knew no formula or mathematical expression; and is told that many sorts of expressions are used besides those he met in elementary algebra. Before he can make much headway it will be necessary to become familiar with various other functions besides simple powers, fractions, sums, etc.; and it will be helpful to consider, first, the way in which the sides and angles of a triangle are related.

The actual definition of the trigonometric functions and solutions of triangles is prefaced, however, by some little use of a protractor. With it the student solves roughly such triangles as arise in surveying or statics. This is nothing more mysterious than drawing a figure to scale and reading off the required parts. This construction work not only makes the student feel that he has a ready solution or check in any such problem, but also helps him to be clear about the application of trigonometry to these subjects. Moreover, before he has any trigonometric functions to think about, it is a good time for him to get the few principles of force-resultants, which he needs to know for the problems in statics. Any one can quickly master such problems as the graphical analysis of a simple framed structure composed of a few triangles. But, of course, we must lead him by natural steps from the parallelogram of forces to such analysis.

At this stage only four functions are introduced, and three-place values are used at first, checked by the protractor to emphasize their meaning. But larger tables are soon used which make the student feel the need of easier methods of calculating, in spite of various arithmetical devices and short cuts which are

shown him. In passing be it said that some oblique triangles are solved by dissecting them or by using the sine law and cosine law.

The work on logarithms is prefaced by use of the notation 2.417×10^{-8}, etc., so common in scientific work. This is both useful in itself and helpful in making clear the treatment of characteristics of logarithms. No rule for characteristics is needed if the student gets to thinking of every number as expressible in this "scientific notation." This saves some time and trouble, and illustrates one method by which much time is saved all through the course: viz., teaching *processes*, instead of rules which take time to learn and are likely to become mechanical. Another illustration of this is the matter of interpolation; in connection with the first graphical work the student learns to interpolate in any table by a method of proportional parts. Consequently he does not need later to be taught how to interpolate in trigonometric, logarithmic, or other numerical tables. A hint suffices to show him the short-cuts after he has had some practice in using the old general process. Of course, no base but 10 is even mentioned until all the actual computing and applications have been covered, which include formulas of physics, engineering, geometry, trigonometry, interest, etc. And it is interesting that students themselves prove the laws for any base, having seen their nature clearly while working with logarithms as exponents of 10.

The student is now ready to differentiate $\log v$ and e^v, to perform integrations involving these, and to solve problems on the "compound interest law." Here he learns also to differentiate products and quotients, which might have been taken up earlier, but at the risk of confusion. In the first calculus work the student had practically nothing in the way of formulas to think about; consequently his whole attention could be given to methods of applying the calculus, and to the meaning of the processes. Each time he learns to differentiate a new type of function, he makes applications which use all the old principles. Thus, through the year, there is frequent review of the underlying principles of rates, acceleration, calculation of areas, work, volumes, differential approximations, etc.

This is followed by further graphical work, but this time from the standpoint of coördinate geometry, while heretofore it has been based merely upon the representation of a function. Starting with the plotting of points, fixed or moving, the student sees how to study the motion of a point by its parametric equations, $x = f_1(t), y = f_2(t)$; finding the velocity and acceleration at any point, and the length of the path traveled. In particular, he deals with the motion of a projectile, both when its equations are given and when he must find them by integration. Next, by seeking a sure test as to whether a given point lies on a certain line or curve, the student sees the relation between a curve and its equation. He then derives the equations of various loci, proves analytically some simple theorems in the geometry of triangles and thus comes to see that algebraic methods may be systematically employed to study geometry. This is further emphasized by the discovery of some properties of conics (previously unknown to him) from their equations which he has set up. The idea of sliding a curve helps him to recognize the locus of any quadratic which lacks the xy term. Applica-

tions of the conics are pointed out here. This work is concluded by using ordinary graph paper to discover physical laws of the linear type, and logarithmic and semi-logarithmic paper to discover some other laws.

This is followed by work on the solution of equations, some of which, by the way, was included in the first graphical work. At this stage, the student is able to use differential approximations to advantage, and the idea of sliding a curve leads him easily to Horner's method. Of course, he also learns to find rational roots exactly by trial. The geometrical interpretation of simultaneous equations leads to some remarks on the fascinating topic of n-dimensional space.

The work on polar coördinates includes the plotting of fixed and moving points with special reference to circular motion, both uniform and non-uniform. In connection with this are studied the relation of an arc to its central angle, radians, estimates involving small angles, simple harmonic motion and the differentiation of the sine and cosine. The trigonometric functions are now defined for angles of any size, their graphs are drawn and their relations to acute angles are studied. Applications are made to alternating currents, parametric equations, etc.

The work on trigonometric analysis begins with the primary relations among the functions of a single angle (which the student has never yet heard of), and the uses of these relations, together with the double angle formulas, in differentiation and integration, including the method of rationalizing an algebraic integral by a trigonometric substitution. The usual formulas are developed for the direct logarithmic solution of oblique triangles and the addition formulas are proved and applied.

The student is now able to handle some of the fairly complicated integrals which arise in applications of the integral calculus. At the same time he learns to calculate volumes by double integrations in simple cases, and to draw plane sections of simple surfaces.

This is followed by some work on progressions and series with applications to interest and annuities, computation of logarithms and trigonometric functions, and integration by expansion into series. Some remarks on imaginary logarithms and Fourier's series are usually made at this point (to the student's interest, if not to his profound understanding!).

In conclusion, some work is usually done on elementary permutations and combinations and simple problems of chance. A brief introduction to the theory of probability lets the student use the method of least squares in the simplest case.

The course as given takes four hours a week through the year. The ground could be covered by lectures in less time; but with so wide a range of topics, it is important to allow time for considerable practice in class. Some time is saved by usually taking up a new topic before leaving the old,—that is, including review problems in nearly every assignment. The closeness with which successive topics are correlated and the frequency of contact with earlier topics make it possible for students to assimilate the material pretty well in spite of the rapidity

of their progress. Concise summaries in mimeographed form help greatly in fixing the grasp of the topics, especially as no text is available which covers any large part of the course from the same standpoint.

The treatment of topics is unconventional in that it keeps their practical uses steadily to the fore, and admits only topics, some of whose uses can be exhibited when the topic is studied. This gives students constant practice in applying their mathematics and in analyzing problems. It also creates considerable enthusiasm for the subject. Students see something of the meaning of mathematical work, and they do a very large amount of work very cheerfully—one may almost say eagerly. (Approximately one thousand exercises are worked during the year, many of which are fairly substantial problems.)

A word concerning the courses which follow. Naturally they must be modified somewhat. The second year is devoted to a systematic course in calculus, but several of the usual preliminaries, which were omitted from the first course, are treated incidentally as needed,—*e. g.*, elements of solid analytics before taking up multiple integration. Also, trigonometric analysis is reviewed in detail before differentiating trigonometric functions. Since the students are already familiar with the elements and general principles, they can be given an excellent grasp of the subject in this year's work. Besides covering a very full text-book treatment, they have time for some use of imaginaries in trigonometric reductions, for elementary problems on Fourier's series and calculus of variations, and for some practice in formulating as well as solving practical differential equations.

The more theoretical parts of college algebra and analytic geometry are postponed to the junior year when they can be dealt with adequately in connection with modern developments in algebra and geometry. Thus it is possible to devote the junior and senior years to mathematics of a fairly advanced type.

It seems to the writer that it would be perfectly feasible to give the freshman course outlined above in the fourth year of high schools and academies wherever there are strong teachers who know the subject well. Even then students would not be starting the study of these topics as young here as abroad. It is not hard to think of advantages which would result from their earlier introduction here.

LINKAGES.

By DICKSON H. LEAVENS, College of Yale in China, Changsha.

The subject of linkages is somewhat removed from the main lines of mathematics, but although it may not be of much importance in their development, it has considerable intrinsic interest. The problem that gave rise to the study of link-motions, that of making a point move in a straight line, is of significance theoretically in furnishing a method of drawing a straight line without begging the question as we do when we copy by means of a ruler a line already made.[1]

[1] See A. B. Kempe, *How to Draw a Straight Line*, London, Macmillan, 1877, which also gives a very clear and simple account of linkages in general. It is perhaps the best introduction to the subject, and its footnotes give references to the original papers up to its date.

This problem is also of practical importance in machinery, where it is necessary to produce rectilinear motion of certain parts. The history of the subject, too, is interesting, three or four great minds have contributed to it, and some remarkable results have been obtained. Finally, it sometimes furnishes a striking way of illustrating graphically results obtained analytically.

A plane linkage may be defined as a system of bars (or more generally, plane figures) pivoted together so that they move in the same plane (*i. e.*, in the case of material links, in close parallel planes). A *complete* linkage will have an even number of bars, so connected that there is only one degree of freedom of motion. A *linkwork* consists of an odd number of bars, pivoted at two points to a fixed plane.[1] If we consider the fixed plane as a link, we have a linkage, so that word is often used loosely instead of linkwork.

Leaving out of consideration the simple and familiar pantograph, invented early in the seventeenth century by C. Scheiner, we may say that the subject had its origin with James Watt. In his improvements on the steam-engine he found the need of guiding the piston rod in a straight line. This rectilinear motion is now effected by having the cross-head slide on smooth-planed guides, but this could not be done with sufficient mechanical perfection in his day, and some method with less friction was required.

In 1784 Watt patented his so-called "parallel motion." In a letter[2] to his son, written twenty-four years later, he says: "The idea originated in this manner. On finding double chains, or racks and sectors, very inconvenient for communicating the motion of the piston-rod to the angular motion of the working-beam, I set to work to see whether I could contrive some means of performing the same from motions turning upon centers, and after some time it occurred to me that AB and CD (Fig. 1),[3] being two equal radii revolving on the centers B and

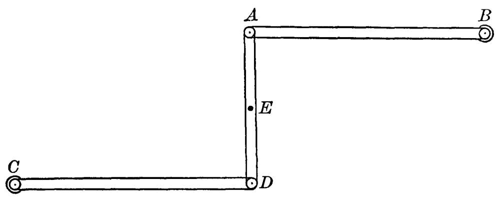

FIG. 1.

C, and connected together by a rod AD, and moving together through arches of certain lengths, the variations from the straight line would be nearly equal and opposite, and that the point E would describe a curve nearly straight, . . .

[1] For these definitions and for future references to Sylvester, see *On Recent Discoveries in the Mechanical Conversion of Motion*, in Sylvester, *Collected Works*, Cambridge Univ. Press, 1909, III, pp. 7–25. This paper is one of the most interesting on the subject.

[2] J. P. Muirhead, *The Life of James Watt*, New York, Appleton, 1859, p. 242.

[3] The properties of linkages are much more vividly brought out, both for individual study and for demonstration to a class, by working models than by mere diagrams. Such models are very easily made, after a little practice, with the links of moderately stiff cardboard, pivoted together with eyelets by means of an eyelet punch and set.

and from this the construction, afterward called the parallel motion, was derived. . . . Though I am not over anxious after fame, yet I am more proud of the parallel motion than of any other invention I have ever made." The whole curve described by E is a figure eight sextic, with the two parts through the node deviating very slightly from a straight line.

Watt applied this to his engines, using the walking-beam for one of the fixed arms, and adding two more links in a pantograph arrangement so that a second point was made to move in a straight line parallel to that described by E. One of these points was attached to the piston-rod and the other to the air-pump rod.[1] It is probably this arrangement that was the origin of the name "parallel motion," which has been carried over, though obviously a misnomer, to the simpler form and to all other linkworks designed to give rectilinear motion to a point.

There does not seem to have been any scientific investigation of the subject during the first half of the nineteenth century. The first mathematician to take it up was Tchebychef (variously spelled as romanized in different countries), professor in the University of St. Petersburg. He became interested in it while giving a course in applied mechanics, and in 1852 took a trip[2] to western Europe, especially to visit factories, see different types of mechanism, and most of all to study the Watt parallel motion. In England he hunted up Watt's original machines,[3] wherever he could find them, in order to ascertain the lengths of the various parts. On his return, he studied the question of getting the closest possible approximation to rectilinear motion by linkwork. This led to an analytical problem which he treated skilfully and at great length, inventing new methods of analysis which are beyond the scope of this paper. His results enabled him to discuss the Watt movement, and to devise new ones, all of them only approximately straight line motions, but in every case accompanied by formulas showing the amount of deviation from a straight line as a function of the arbitrary constants of the linkwork. This deviation, by a proper choice of the constants, can be made very small, well within the limits of mechanical accuracy in the construction, so that for practical purposes they are quite as useful as if theoretically exact.

Tchebychef had considered the problem of obtaining exact rectilinear motion, and had finally come to the conclusion that it was impossible, although he did not succeed in proving this. In 1871, however, one of his own students, Lipkin, discovered a seven-bar linkwork which would describe an exactly straight line. The Russian government awarded him a prize, and then it transpired that identically the same device had been discovered by a French officer, Peaucellier,

[1] See the *Century Dictionary*, 1911, p. 4278, under "parallel" for picture of application to walking-beam.

[2] For his own interesting account of the trip, see *Œuvres de P. L. Tchebychef* [in French], St. Petersburg, 1899, Vol. II, pp. vii–xviii. The various papers giving his work on linkages and the functions associated with them will be found in the two volumes. A brief general survey of his life and work, and a special account of his results on linkages, are contained in *P. L. Tschebyschef und seine wissenschaftlichen Leistungen*, Wassilief and Delaunay, Teubner, 1900.

[3] A very interesting collection of original engines and models made by James Watt and his workmen may now be seen in the Science Division of the South Kensington Museum, London.

in 1864, and announced, but not described, in the *Nouvelles Annales*[1] of that year, without attracting much attention. Peaucellier's priority and Lipkin's independence were both fully established.

This seven-bar linkwork (Fig. 2) consists essentially of a six-bar linkage, called the Peaucellier cell, and an extra link. The cell is composed of a rhombus, P_1MP_2M', and two equal bars, OM, OM', pivoted to opposite vertices and pivoted

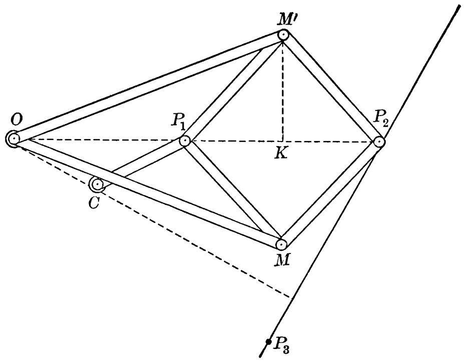

FIG. 2.

together at O. By symmetry the three points O, P_1, P_2 (respectively called the fulcrum, first pole, and second pole) are always collinear, and by simple geometry:

$$\overline{OP}_1\cdot\overline{OP}_2 = (\overline{OK} - \overline{P_1K})\cdot(\overline{OK} + \overline{P_1K}) = \overline{OK}^2 - \overline{P_1K}^2$$

$$= (\overline{OM}^2 - \overline{KM}^2) - (\overline{P_1M}^2 - \overline{KM}^2) = \overline{OM}^2 - \overline{P_1M}^2, \text{ a constant.}$$

That is, the product of the distances of the two poles from the fulcrum is constant. Thus, keeping O fixed, if P_1 describes any curve, P_2 will describe the inverse curve. It is a well-known fact, which may be shown by elementary geometry, that the inverse of a circle is in general a circle, but when the center of inversion lies on the original circle, the inverse is a straight line. Hence by constraining P_1 to move in a circle through O, which may be done by pivoting O to a fixed plane, and adding another link, CP_1 pivoted at C so that $CP_1 = OC$, we may make P_2 move in a straight line P_2P_3, perpendicular to OC. Or, by making $CP_1 \neq OC$, we may make P_2 move in a circular arc of any desired radius, by properly adjusting the constants.

This beautiful device aroused much interest in the study of linkages in general, and during the five years beginning 1874 more than one hundred[2] papers on the subject appeared in the mathematical journals. Various mathematical applications and generalizations of the Peaucellier apparatus were made. In mechanism, comparatively little use seems to have been made of it, for other methods of

[1] 2e série, III, p. 414.

[2] Liguine, *Liste de travaux sur les systèmes articulées*, *Bull. des Sci. Math.*, 2e série, VII, pp. 145–160, gives 150 titles up to 1882. For later bibliography see *Royal Society Catalogue of Scientific Papers, 1800–1900, Subject Index*, Vol. II, *Mechanics*, pp. 84–86.

getting rectilinear motion are more convenient in practice. Many special linkworks have been constructed to describe different curves, such as conics, cardioids, limaçons, lemniscates, cissoids, etc. The simplest conicograph contains seven bars. Linkages have been applied to complex variables and elliptic functions, and to realizing numerous algebraic transformations.[1]

The study of the curves described by a linkwork can most readily be made by the use of polar coördinates, which may later be transformed into Cartesian.[2] In general, a three-bar linkwork describes a sextic curve, and in general the curves described by any linkwork turn out to be algebraic. This and its converse were stated by Peaucellier[3] from general reasoning, which did not satisfy Sylvester, who outlined a proof. It remained for Kempe to prove explicitly: "A linkwork can be found to describe any given algebraic curve." There is no space here to outline his simple and elegant proof.[4] The method is much more complicated than most of the special linkworks devised for important curves, but it is of theoretical value. The opposite theorem is also true, that no transcendental curve can be described by a linkwork. These results definitely show us the large possibilities of linkworks, and also their limitations.

THE PROBABLE RANK IN A LARGE CLASS OF A STUDENT OF GIVEN RANK IN A SMALL CLASS.

By L. D. AMES, University of Missouri.

In a certain statistical investigation it was desirable to compare students in different classes on the basis of their ranks in their respective classes. To reduce the two rankings to a common unit we seek to know the probable rank in a very large class of a student who ranks kth in a class of n students.

For example, we may wish to know how a given class of students who have had a certain definite type of training compare in their subsequent work with the average of students with whom they may compete. We follow these particular students into various other classes and find, for example, that one of them ranks second in a class of nine, another ranks sixth in a class of ten, etc. We wish to assign a numerical value to these rankings. Does the second student in a class of four probably rank, other things being equal, on a par with the twenty-second student in a class of forty-four?

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[1] See Emch, *An Introduction to Projective Geometry and Its Applications*, Wiley, 1905, for a chapter on linkage transformations and references to original papers.

[2] For examples of the methods of attack, see F. Dingeldey, *Über die Erzeugung von Kurven vierter Ordnung durch Bewegungsmechanismen*, Teubner, 1885, Chapter III. Also Königs, *Leçons de cinématique*, Paris, 1897, Chap. XI.

[3] *Note sur une question de géometrie de compas, Nouvelles Annales*, 2e série, XII, p. 71, 1873. This, by the way, is Peaucellier's first published description of his invention.

[4] *On a General Method of Describing Curves of the nth Degree by a Linkwork. Proc. of London Math. Soc.*, 1876, VI, pp. 213–216. Also given in Königs, *op. cit.*, pp. 269–273.

To answer this question we suppose that a very large number of students are equally distributed along a line of given length. A class of n students is picked at random from this large number. The kth student of this class, counting from one end, is noted, and his distance from that end. This is repeated a large number of times and the distances averaged. What is the probable value of this average?

Clearly the second student in a class of three, and the fifth student in a class of nine stand at the middle of their respective classes, or at 1/2 on a unit scale. The result reached below agrees with this conclusion. In a class of three the first student would be ranked as 1/4, the second as 2/4, the third as 3/4. In general, if a scale of any convenient length is divided into equal parts one greater in number than the number of students in the class, and the students are ranked on the scale at the points of division in the order of their rank in the class, the end points of the scale not being used, then their distances from one end will represent their probable ranks in a very large class.

To avoid irrelevant assumptions we state the problem in abstract form. If n points be independently chosen at random on the interval from $x = 0$ to $x = 1$, find the probable distance from $x = 0$ to the kth point counting from that end. By the words "chosen at random" we mean that the probability that any specified point will be taken from any interval is the same as the probability that it will be taken from any equal interval.

Let us think of the points as chosen successively. The probability that the first point chosen will be chosen from the interval from x to $x + \Delta x$ is precisely Δx. The probability that the next $k - 1$ points chosen will be chosen from the interval from 0 to x is x^{k-1}. The probability that the last $n - k$ points chosen will be chosen from the interval from x to 1 is $(1 - x)^{n-k}$. The probability that all of these things will happen together is the product of their separate probabilities. But, as any one of the n points is equally likely to be the kth point from the origin, we must multiply this by n, and as for each one of these cases the number of different possibilities of selecting the points which fall in the interval from 0 to x is the number of combinations of $n - 1$ things taken $k - 1$ at a time, the probability that the kth point from the origin will lie in the interval from x to $x + \Delta x$ is the product of these probabilities, or

$$n \cdot {}_{n-1}C_{k-1} x^{k-1}(1 - x)^{n-k}\Delta x.$$

If now we multiply this probability by x, the distance from the origin to a point of the interval, add these terms for all the intervals, take the limit as Δx approaches zero and divide by the total probability which is 1, we have as the probable average distance from the origin to the kth point counting from the origin the definite integral

$$n \cdot {}_{n-1}C_{k-1} \int_{x=0}^{x=1} x^k(1 - x)^{n-k}dx.$$

Successive integration by parts gives the probable distance as $k/(n + 1)$.

If both classes are supposed finite, we may ask what is the probable rank, K, in a class of N students of the kth student in a class of n students. We have $K/(N+1) = k/(n+1)$, or $K = k(N+1)/(n+1)$. Thus, for example, the second student in a class of four, according to the above assumptions would stand at a distance of 2/5 from the origin, and would rank the same as the eighteenth student in a class of 44.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY, University of Minnesota.

The Development of the Arabic Numerals in Europe Exhibited in Sixty-four Tables. By G. F. HILL. Oxford, Clarendon Press, 1915.

In *Archæologia,* Vol. LXII, Mr. G. F. Hill, keeper of the department of medals and coins in the British Museum, published a series of fifty-one tables of Hindu-Arabic numerals as they appeared in Mss. and on monuments, coins, seals, medals, brasses, and paintings, as well as a few forms from printed works. This article has been of great value to all who are interested in the exact dating of mediæval manuscripts, for the work furnishes, where numerals are used, definite checks upon the time when a manuscript was written. Further and more particularly, the work is of interest to students of the history of science for it shows in graphical form the slow but sure progress of the Hindu arithmetic through Europe. Hearty welcome, then, is given to the present publication of the Oxford Press, in which appear not only the tables of the earlier work, but also additional tables of forms more recently located.

In general the forms included are only those which antedate 1500 A.D., for by that time the new arithmetic had become well-nigh universal. Parenthetically it is of interest to note, however, that as late as 1540, Köbel, in Germany, published an arithmetic wholly, except paging, in Roman numerals.

The earliest European forms are doubtless those found in the *Codex Vigilanus,* written 976 A.D. in the monastery of Albelda near Logrono in Spain. A second Spanish manuscript of about the same date, not described by Mr. Hill, also contains similar forms, and facsimiles. Both are to appear in the next issue of Professor John M. Burnam's *Palæographia iberica.* The numerals appear to have been known in Syria in 662 A.D., for a Syrian Bishop, Severus Sebokt, of a monastery on the Euphrates, refers in a work of that time to the "easy method of their, *i. e.,* the Hindus', calculations and of their computations which surpasses words. I mean that made with nine symbols," referring certainly to the Hindu system of numerals with the zero.

Interesting also is the fact that Mr. Hill finds that the forms which afford the best criteria for dating are those for 2, 4, and 7, while 5, "the most freakish of all figures," comes next. In probably the earliest translation of an Arabic treatise on the Hindu arithmetic, the *Algoritmi de numero Indorum* (published by Bon-

compagni in 1857), a translation made quite certainly in the early part of the twelfth century, the unknown writer states that there is some diversity among men in writing the figures for 5, 6, 7, and 8. This may even have been in Al-Khowarizmi's lost original treatise on arithmetic; no forms are found at this point in the unique Cambridge MS. of the translation. In passing we may note that the 2 written in form like a 7 is indicative of writing earlier than about 1,350 while a three which resembles a printed *r* (small *r*) is characteristic of the twelfth or early thirteenth century.

Mr. Hill has previously established a reputation in the field of numismatics, particularly Greek, and in Greek history. In the present treatise Mr. Hill has rendered a real service to the history of science by his patient and long-continued search for early appearances of the Hindu-Arabic numerals in Europe and by their publication in this convenient form.

The price of the book is $1.00; and it may be obtained from the Oxford University Press, 29 W. 32d St., New York City.

LOUIS C. KARPINSKI.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

PROBLEMS FOR SOLUTION.

ALGEBRA.

445. Proposed by S. A. JOFFE, New York City.

Sum the series

$$\binom{cn}{a} - \binom{a}{1}\binom{cn-c}{a} + \binom{a}{2}\binom{cn-2c}{a} - \cdots + (-1)^a\binom{cn-ca}{a}.$$

446. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Solve the equations

$$x^2(y-z) = l^2(m-n), \qquad y^2(z-x) = m^2(n-l), \qquad z^2(x-y) = n^2(l-m).$$

447. Proposed by ELIJAH SWIFT, University of Vermont.

(*a*) A method is sought of forming an equation such that the first k figures of some root shall be given numbers. For example, form a cubic equation such that one of its roots is $1.918+$.

(*b*) Of all the equations suggested in (*a*), determine that one for which the sum of the absolute values of the coefficients is least.

GEOMETRY.

476. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Show that the locus of the middle points of a set of parallel chords intercepted between an hyperbola and its conjugate is $4b_1^2x^2y^2 - 4a_1^2y^4 = a_1^2b_1^4$.

Ashton's *Analytical Geometry*, p. 194, Prob. 34.

477. Proposed by N. P. PANDYA, Sojitra, India.

A village A_0 is equidistant from ten villages, $A_1, A_2, A_3, \cdots, A_{10}$. The distances, A_1A_2, A_2A_3, $\cdots$, A_9A_{10}, $A_{10}A_1$, are in arithmetic progression. A person, starting from A_0, has to go to four of these villages consecutively, and has then to return to A_0. What four villages should he select so that the total distance traveled by him may be a minimum?

478. Proposed by J. A. CAMPBELL, St. Johnsbury, Vt.

Construct a triangle having given the difference of the segments of the base formed by the foot of the perpendicular from the vertical angle, the difference of the base angles, and the sum of the three sides.

CALCULUS.

397. Proposed by C. N. SCHMALL, New York City.

On the radii vectores of one loop of the lemniscate, $\rho^2 = a^2 \cos 2\theta$, as diameters, circles are described, passing through the pole. Find the locus of their points of intersection and show that its area is twice that of the loop.

398. Proposed by V. M. SPUNAR, Chicago, Ill.

Solve

$$2\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = 0.$$

Proposed by EMMA GIBSON, Drury College.

Solve the differential equation,

$$(x^3y^3 + x^2y^2 + xy + 1)y + (x^3y^3 - x^2y^2 - xy + 1)\,x\frac{dy}{dx} = 0.$$

From Forsyth's *Differential Equations*, p. 48, Ex. 1, XXIV.

399. Proposed by B. J. BROWN, Victor, Colorado.

A cow is tethered by a perfectly smooth rope, a slip noose in the rope being thrown over a large square post. If the cow pulls the rope taut in the direction shown in the figure, at what angle will the rope leave the post?

From Granville's *Diff. and Int. Calculus*, p. 120, Prob. 55.

MECHANICS.

317. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Show that the maximum area contained between the path of a projectile and the horizontal line is $\frac{v^2\sqrt{3}}{8g^2}$, where v is the velocity of projection.

318. Proposed by C. N. SCHMALL, New York City.

Given an inclined plane making an angle ϕ with the horizontal. A perfectly elastic ball is projected upward at an angle ψ with the inclined plane, so as to ascend it by bounds. Show that as the ball rebounds for the nth time, the angle of inclination of its path to the plane is

$$\tan^{-1}\left(\frac{\tan\psi}{1 - 2n\tan\phi\tan\psi}\right);$$

and if it rebounds vertically upward, then $\cot\psi = (2n + 1)\tan\phi$.

NUMBER THEORY.

235. Proposed by W. D. CAIRNS, Oberlin College.

Prove that $n = 1$ is the only positive integer for which $n^4 + 4$ is a prime.

236. Proposed by V. M. SPUNAR, Chicago, Illinois.

Find integral values of x, y, z, such that

$$xy + z = \square, \qquad yz + x = \square, \qquad \text{and} \qquad xz + y = \square.$$

SOLUTIONS OF PROBLEMS.

ALGEBRA.

434. Proposed by S. A. JOFFE, New York City.

Express the "difference of zero" $\Delta^n O^{n+1}$ in the form: $c_1(n+2)! - c_2(n+1)!$, where c_1 and c_2 are numerical coefficients independent of n.

Solution by the Proposer.

From the theory of Finite Differences we know that

$$\Delta^n O^i = n(\Delta^n O^{i-1} + \Delta^{n-1} O^{i-1}).$$

If we put here $i = n + 1$, we obtain

$$\Delta^n O^{n+1} = n(\Delta^n O^n + \Delta^{n-1} O^n),$$

which, after division by $n!$, gives

$$\frac{\Delta^n O^{n+1}}{n!} = \frac{\Delta^n O^n}{(n-1)!} + \frac{\Delta^{n-1} O^n}{(n-1)!},$$

or, since $\Delta^n O^n = n!$,

$$\frac{\Delta^n O^{n+1}}{n!} - \frac{\Delta^{n-1} O^n}{(n-1)!} = n. \tag{1}$$

Replacing n in equation (1) successively by $n-1$, $n-2$, $\cdots$, 1, we get the following $n-1$ similar equations:

$$\frac{\Delta^{n-1} O^n}{(n-1)!} - \frac{\Delta^{n-2} O^{n-1}}{(n-2)!} = n-1; \quad \frac{\Delta^{n-2} O^{n-1}}{(n-2)!} - \frac{\Delta^{n-3} O^{n-2}}{(n-3)!} = n-2; \quad \cdots \quad \frac{\Delta^2 O^3}{2} - \frac{\Delta O^2}{1} = 2; \quad \frac{\Delta O^2}{1} = 1.$$

Adding together the preceding n equations, including (1), we obtain, after cancellation,

$$\frac{\Delta^n O^{n+1}}{n!} = n + (n-1) + (n-2) + \cdots + 1 = \frac{n(n+1)}{2},$$

which may be written in this form:

$$\Delta^n O^{n+1} = \frac{n}{2}(n+1)! \tag{2}$$

Finally, since $\frac{n}{2} = \frac{1}{2}(n+2) - 1$, equation (2) may be expressed as follows:

$$\Delta^n O^{n+1} = \frac{1}{2}(n+2)! - (n+1)! \tag{3}$$

which is the form desired.

435. Proposed by C. N. SCHMALL, New York City.

Show that $(e-1) - \frac{1}{2}(e-1)^2 + \frac{1}{3}(e-1)^3 - \cdots = 1$, where e is the Naperian base of logarithms.

Solution by Geo. W. Hartwell, Hamline University.

This equation results from substituting $(e-1)$ for x in the series,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots.$$

This, however, is not a permissible substitution, since the logarithmic series

converges only when $x \leqq 1$. Moreover, the given series is not convergent, since

$$\operatorname*{Lt}_{n \doteq \infty} \frac{(e-1)^n}{n} \neq 0,$$

$e - 1$ being greater than unity.

Also solved by HORACE OLSON, C. E. FLANAGAN, J. W. CLAWSON, and H. S. UHLER.

GEOMETRY.

449. Proposed by H. E. TREFETHEN, Colby College.

Find a tetrahedron with the face angles at one vertex in arithmetical progression and its six edges expressed in positive integers.

SOLUTION BY THE PROPOSER.

Let the angles be $A + B$, A, $A - B$; the lateral edges x, y, z; the base edges a, b, c; so that $a^2 = x^2 + y^2 - 2xy \cos(A + B)$, $b^2 = x^2 + z^2 - 2xz \cos A$, $c^2 = y^2 + z^2 - 2yz \cos(A - B)$. If, for brevity, we put $1 + \cos(A + B) = P/2$, $1 + \cos A = Q/2$, and $1 + \cos(A - B) = R/2$, we may write

$$a^2 = (x + y)^2 - Pxy = (x + y - p)^2 \text{ or } Pxy - 2py = 2px - p^2, \tag{1}$$

$$b^2 = (x + z)^2 - Qxz = (x + z - q)^2 \qquad Qxz - 2qz = 2qx - q^2, \tag{2}$$

$$c^2 = (y + z)^2 - Ryz = (y + z - r)^2 \qquad Ryz - 2ry = 2rz - r^2. \tag{3}$$

Eliminating y and z from (1), (2), (3), and arranging we have

$$(2px - p^2)[q(4r - Rq) + 2(Rq - Qr)x] = r(Px - 2p)[2q(r - q) + (4q - Qr)x]. \tag{4}$$

If we put the coefficient of $x^2 = 0$, then

$$p = \frac{Pr(4q - Qr)}{4(Rq - Qr)} \tag{5}$$

and also

$$2x = \frac{p^2q(4r - Rq) + 4pqr(q - r)}{p(8qr - Qr^2 - Rq^2) - p^2(Rq - Qr) + Pqr(q - r)}. \tag{6}$$

We may now assign values to A and B and thus determine P, Q, R. If values are assigned to q and r also, p is defined by (5), and then x may be found from (6), y and z from (1), (2), (3) and finally a, b, c also.

Thus if $A = 90°$ and $B = \text{arc-sin } 1/3$, then $P = 4/3$, $Q = 2$, $R = 8/3$; and if also $q = r = 1$, then $p = 1$, $x = 1/4$, $y = 3/10$, $z = 1/3$. Reducing the values of x, y, z to a common denominator and rejecting it, we have in integers $x = 15$, $y = 18$, $z = 20$, and consequently $a = 27$, $b = 25$, $c = 22$. Or since the equations are symmetrical we may use the reciprocals $x = 4$, $y = 10/3$, $z = 3$, whence in integers $x = 12$, $y = 10$, $z = 9$, and then $a = 18$, $b = 15$, $c = 11$. Again if $\sin B = 1/2$, the angles are 120°, 90°, and 60°; $x = 9$, $y = 15$, $z = 40$; $a = 21$, $b = 41$, $c = 35$. If $\sin B = 2/3$, we find $x = 1{,}092$, $y = 416$, $z = 81$; $a = 2{,}804$, $b = 2{,}185$, $c = 367$.

By varying q, r and the $\sin B$, other sets of numbers may be found ad libitum.

464. Proposed by FRANK R. MORRIS, Glendale, Calif.

The sum of the hypotenuse and one side of a right triangle is 100 feet. A point on the hypotenuse is 10 feet from each of the sides. Find the length of the hypotenuse correct to the third decimal place.

SOLUTION BY W. W. BURTON, Mercer University, Macon, Ga.

Let $BC = x$. Then $AB = 100 - x$ and $BD = x - 10$. The right triangles ACB and PDB are similar. (Their sides are respectively parallel.) Therefore

$$AB : BC = PB : BD \text{ or } (100 - x) : x = PB : (x - 10),$$

and

$$PB = \frac{(x - 10)(100 - x)}{x}.$$

Again

$$\overline{PB}^2 = \overline{PD}^2 + \overline{BD}^2 \text{ or } \overline{PB}^2 = 10^2 + (x - 10)^2 \text{ or } x^2 - 20x + 200.$$

Therefore

$$PB = \sqrt{x^2 - 20x + 200},$$

and

$$\frac{(x - 10)(100 - x)}{x} = \sqrt{x^2 - 20x + 200}.$$

Reducing, we get the cubic,

$$2x^3 - 139x^2 + 2200x - 10000 = 0.$$

Applying Descartes' rule of signs we find that all of the roots of the above cubic are positive. Applying Sturm's theorem we find that the roots are situated as follows: one root between 9 and 10; one root between 11 and 12; and one root between 49 and 50.

The root between 9 and 10 must be discarded, as it is impossible in our problem. Applying Horner's method for incommensurable roots, we find the roots to be 11.282 and 49.212 correct to three decimal places. Therefore $BC = 11.282$ or 49.212. Hence, the hypotenuse can be *88.718* ft. or *50.788* ft.

Also solved by HERBERT N. CARLETON, J. W. CLAWSON, HORACE OLSON, M. HELEN KELLEY, C. E. FLANAGAN, W. E. WHITFORD, G. H. HARTWELL, and NATHAN ALTSHILLER.

465. Proposed by ROGER A. JOHNSON, Western Reserve University.

Let C be a fixed circle, A a point outside it. Let AT and AT' be the tangents from A to the circle, touching the latter at T and T'. Let two secants be drawn through A, cutting the circle at P, Q and R, S respectively. Let PR and QS meet at X, PS and QR meet at Y. Prove by elementary methods that for all positions of the secants, X and Y lie on the line TT'.

SOLUTION BY J. W. CLAWSON, Collegeville, Pa.

I. $PQSR$ is a quadrangle, having X, Y, A for its diagonal points. Hence $X(ARYS)$ is a harmonic pencil. Then, if XY cuts ARS at Z, $(ARZS)$ is a harmonic range.

Now TT' is the polar of A with respect to the circle whose center is C. Hence, if TT' cuts ARS at Z', $(ARZ'S)$ is a harmonic range.

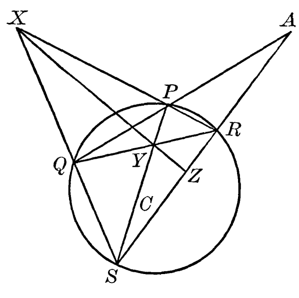

FIG. 1.

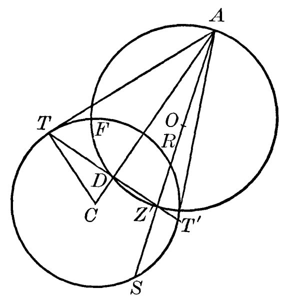

FIG. 2.

Therefore, the points Z and Z' coincide and XY and TT' cut the line ARS at the same point. Similarly, it can be shown that XY and TT' cut the line APQ at the same point. Hence the lines XY and TT' coincide.

[The exercise follows at once from the theorem: If a system of conics circumscribe a given quadrangle, the diagonal point triangle is a self-conjugate triangle w. r. t. each conic of the system. (Durell, Plane Geometry for Advanced Students, Part ii, p. 110.)]

If the above proof is not considered "elementary," a more detailed proof, avoiding the use of the terms "harmonic range" and "polar" is as follows:

II. Extend XY to meet ARS at Z.

Then, since XZ, SP, RQ are concurrent at Y, by Ceva's Theorem,

$$\frac{SZ}{ZR}\cdot\frac{RP}{PX}\cdot\frac{XQ}{QS}=1.$$

Again, since APQ is a transversal cutting the sides of the triangle XSR, by Menelaus' Theorem,

$$\frac{SA}{AR}\cdot\frac{RP}{PX}\cdot\frac{XQ}{QS}=-1.$$

Comparing these results, we see that

$$\frac{SZ}{ZR}=-\frac{SA}{AR}.$$

Again, let TT' cut SR at Z'. From A draw AD perpendicular to TT'. On AZ' as diameter draw a circle. This circle passes through D. Bisect AZ' at O. Let one of the points of intersection of the two circles be F.

Now, $\triangle$s ACT, TCD are similar. Hence $CT^2 = CD\cdot CA$, *i. e.*, $CF^2 = CD\cdot CA$. Therefore, CF is a tangent to circle O. Hence, $\angle CFO$ is a right angle. Hence, OF is a tangent to circle C. Hence, $OF^2 = OR\cdot OS$, *i. e.*, $OZ'^2 = OR\cdot OS$, *i. e.*,

$$\frac{OS}{OZ'}=\frac{OZ'}{OR},$$

i. e.,

$$\frac{OS-OZ'}{OS+OZ'}=\frac{OZ'-OR}{OZ'+OR},$$

i. e.,

$$\frac{Z'S}{AS}=\frac{RZ'}{AR}.$$

Hence,

$$\frac{SZ'}{Z'R}=-\frac{SA}{AR}.$$

Hence, the points Z and Z' coincide. Therefore, the lines XY and TT' intersect ARS at the same point. Similarly it can be shown that XY and TT' intersect APQ at the same point. Hence XY and TT' coincide.

Also solved by N. P. Pandya.

CALCULUS.

378. Proposed by ELBERT H. CLARKE, Purdue University.

The area of the curved surface generated by the revolution about OX of the portion of the curve $y = x^n$ which extends from the origin to the point (1, 1) is given by the formula

$$A = 2\pi\int_0^1 x^n\sqrt{1+n^2x^{2n-2}}\cdot dx.$$

Our geometric intuition would tell us that the limit of this area as n becomes infinite is π. Give a strict analytic proof that

$$\lim_{n\doteq\infty}\int_0^1 x^n\sqrt{1+n^2x^{2n-2}}\cdot dx=\tfrac{1}{2}.$$

Solution by Elijah Swift, University of Vermont.

We give the proof by writing the above integral as the sum of two and showing that the limit of one is zero, of the other, $\frac{1}{2}$. Let k be any value between 0 and 1, *e. g.*, .9. Then

$$\int_0^1 x^n\sqrt{1+n^2x^{2n-2}}\cdot dx=\int_0^k x^n\sqrt{1+n^2x^{2n-2}}\cdot dx+\int_k^1 x^n\sqrt{1+n^2x^{2n-2}}\cdot dx.$$

But the integrand of the first of these two integrals is less than (or equal to) $k^n\sqrt{1+n^2k^{n-2}}$ and since the limit of this is 0, the limit of the integral is zero also. To handle the second integral, write the integrand as

$$nx^{2n-1}\sqrt{1+\frac{1}{n^2x^{2n-2}}}.$$

We may now develop in series and show that the limit of every term after the first is zero, or we may proceed as follows:

$$\int_k^1 nx^{2n-1}\sqrt{1+\frac{1}{n^2x^{2n-2}}}\,dx = \int_k^1 nx^{2n-1}\left\{1+\sqrt{1+\frac{1}{n^2x^{2n-2}}}-1\right\}dx$$

$$= \int_k^1 nx^{2n-1}\cdot dx + \int_k^1 n\cdot x^{2n-1}\cdot \frac{\dfrac{1}{n^2x^{2n-2}}}{\sqrt{1+\dfrac{1}{n^2x^{2n-2}}}+1}\cdot dx,$$

by rationalizing the numerator. The limit of the first integral is $\frac{1}{2}$ and since the integrand in the second may be made as small as we please, the limit of the second integral is zero.

Also solved by J. A. CAPARO and the PROPOSER.

379. Proposed by C. N. SCHMALL, New York City.

Express the equation of the folium, $x^3+y^3=3axy$, in parametric form and find the area of the loop.

(From E. B. Wilson's *Advanced Calculus*, p. 296, ex. 5.)

SOLUTION BY E. B. WILSON, Mass. Institute of Technology.

Let $y=mx$, then

$$x=\frac{3am}{1+m^3},\qquad y=\frac{3am^2}{1+m^3};$$

the loop being described by values of m from 0 to ∞. By the formulas for area as a curvilinear integral

$$A=-\int_{m=0}^{\infty} y\,dx = -\int_0^\infty 9a^2\frac{m^2(1-2m^3)dm}{(1+m^3)^3} = -\int_0^\infty 3a^2\frac{1-2u}{(1+u)^3}du,$$

where $u=m^3$. Then

$$A=-3a^2\left[\frac{2}{1+u}-\frac{3}{2}\frac{1}{(1+u)^2}\right]_0^\infty = \frac{3}{2}a^2.$$

Also solved by ELIJAH SWIFT, C. E. HORNE, WILSON L. MISER, W. C. EELLS, HORACE OLSON, J. A. CAPARO, H. L. AGARD, L. G. WELD, and the PROPOSER.

MECHANICS.

297. Proposed by C. N. SCHMALL, New York City.

A shrapnel shell strikes the ground and then explodes, dispersing its fragments in all directions with a common velocity v. If a be the area of the ground covered by the fragments, and if the dimensions of the shell be neglected, show that $a=\pi v^4/g^2$.

SOLUTION BY HORACE OLSON, Chicago, Illinois.

According to the laws of physics, the range of a projectile on the horizontal plane from which it is thrown is $(2v^2\sin\theta\cos\theta)/g$ or $(v^2\sin 2\theta)/g$, θ being the inclination with the horizontal of the line of projection. This range has a maximum value, v^2/g, when θ is 45°. Therefore the area of the ground covered by the fragments is $\pi v^4/g^2$, the area of a circle of radius v^2/g.

Also solved by A. M. HARDING, J. L. RILEY, and P. PEÑALVER.

298. Proposed by C. N. SCHMALL, New York City.

A person desires to throw a stone so as to strike the greatest possible blow at a point in a smooth vertical wall at a height h above the ground. If his strength is sufficient to throw the stone vertically upwards to a height $2h$, show that he must stand at a distance $2h$ from the wall. (The resistance of the air and the height of the hand are not taken into account.)

SOLUTION BY PAUL CAPRON, United States Naval Academy.

As the wall is smooth, the force of the blow is proportional to the square of the horizontal velocity of the stone, which in turn is the product of the given initial speed of the stone, v_0, and $\cos\alpha$, if α is the angle of elevation at the start. The problem is therefore to make $\cos\alpha$ a maximum, or to make α a minimum. $(0 < \alpha < \pi/2.)$ Let the required distance from the wall be z; then

$$z = v_0 \cos\alpha t, \qquad h = v_0 \sin\alpha t - \tfrac{1}{2}gt^2;$$

whence

$$2hv_0^2 = 2v_0^2 z \tan\alpha - gz^2 \sec^2\alpha.$$

Differentiating,

$$d\alpha(v_0^2 z \sec^2\alpha - gz^2 \sec^2\alpha \tan\alpha) = dz(gz \sec^2\alpha - v_0^2 \tan\alpha).$$

$$\frac{d\alpha}{dz} = 0$$

if

$$gz \sec^2\alpha - v_0^2 \tan\alpha = 0;$$

i. e., if

$$z = \frac{v_0^2}{g} \sin\alpha \cos\alpha.$$

Then

$$2hv_0^2 = \frac{2v_0^4}{g} \sin^2\alpha - \frac{v_0^4}{g} \sin^2\alpha = \frac{v_0^4}{g} \sin^2\alpha;$$

i. e.,

$$\sin^2\alpha = \frac{2gh}{v_0^2},$$

and therefore,

$$z = \sqrt{\frac{2h}{g}(v_0^2 - 2gh)}.$$

According to the given conditions,

$$v_0^2 = 2g \times 2h = 4gh,$$

so that $\alpha = 45°$, $z = 2$h.

The stone is at the highest point of its trajectory when it strikes, whatever the value of v_0.

We may also solve the problem as follows:

If the stone strikes the wall with a velocity v_1, at an angle β with the horizontal, the energy used up by the blow will be $\frac{1}{2}mv_1^2 \cos^2\beta$, where $v_1^2 = v_0^2 - 2gh$, and $v_1 \cos\beta = v_0 \cos\alpha$. v_0 and h being given, it is necessary for the greatest effect that $\beta = 0$. Then $v_1 = v_0 \cos\alpha$, $v_0^2 \sin^2\alpha = 2gh$, $\sin^2\alpha = 2gh/v_0^2$, and since $2hv_0^2 = 2v_0^2 z \tan\alpha - gz^2 \sec^2\alpha$,

$$2h = 2z\sqrt{\frac{2gh}{v_0^2 - 2gh}} - gz^2 \frac{v_0^2}{v_0^2 - 2gh}, \qquad \text{or} \qquad z^2 - 2\sqrt{\frac{2h}{g}(v_0^2 - 2gh)} \cdot z + \frac{2h}{g}(v_0^2 - 2gh) = 0;$$

i. e.,

$$z = \sqrt{\frac{2h}{g}(v_0^2 - 2gh)},$$

as before.

Also solved by MARCUS SKARSTEDT.

QUESTIONS AND DISCUSSIONS.

EDITED BY U. G. MITCHELL, University of Kansas.

27. A certain college wishes to offer twelve hours of mathematics beyond the usual courses in analytical geometry and differential and integral calculus. Considering only the needs of students intending to specialize in pure mathematics, what courses should make up the twelve hours offered?

REPLY BY W. A. HARSHBARGER, Washburn College, Topeka, Kansas.

This subject seems at first glance to be comparatively simple and to admit of a reasonably definite answer. Every teacher of college mathematics doubtless has a pretty clear idea of what subjects a student majoring in pure mathematics should pursue in the last two years of his undergraduate course, and it is quite likely that these foot up to twice twelve hours. The real problem is one of elimination.

Introduction. Let us suppose that the student enters with the ordinary high-school preparation and the ordinary development in mathematical reasoning. The freshman year is devoted to trigonometry and so-called college algebra, three hours being devoted to each subject. The classes are large and made up of students of various grades of ability from superior to microscopic. Also they contain, side by side, students who will not pursue the subject beyond the freshman year, students who expect to take engineering work, and the few, comparatively, who will specialize in pure mathematics. Also, unless my experience is exceptional, there will be various degrees of preparation in the fundamental operations of the subject, from good to that which is really worse than none at all. Consequently the instruction in the freshman year, particularly in the algebra, must, to a considerable extent, be given to strengthening and correcting the preparation. For this reason but little college algebra can be given in the freshman year. A brief, but thorough review of the fundamental operations, surds, exponents, etc.; then a thorough treatment of the quadratic in one and two unknowns, with as full a consideration of the graph as time will permit, will occupy half the term, or probably a little less. The more advanced topics, complex numbers, permutations and combinations, determinants, theory of equations, etc., thus receive somewhat scant treatment.

In the sophomore year I suppose the student to take five hours each of analytic geometry and calculus. Here again the classes are large and of a mixed character. The work in analytic geometry will cover thoroughly the straight line, circle and conic sections, with a brief course in the geometry of three dimensions, always with the object of teaching the analytical method, rather than of teaching facts about the conics themselves.

In the calculus I presuppose a considerable facility in the processes of differentiation and integration, with simple applications and a great many problems. Infinite series, expansion by Taylor's and Maclaurin's theorems, and the definite integral may be supposed to receive careful consideration, but little will be done with partial differentiation.

We have now conducted the class up to the point contemplated by the question and probably have a comparatively small number in line for the next twelve hours in pure mathematics. As I understand the question, the purpose is to select for the next twelve hours those subjects that will best fit the student for graduate work in pure mathematics. An inventory of his work up to this time will surely show a deficiency in algebra. Hence I would have him devote the first three hours to a somewhat advanced course in algebra.

Algebra. The aim of this course is not to teach the higher algebra as that term is generally understood, but to carry the ordinary processes to a stage where their power will be appreciated; to give facility in execution; in short, to make algebra a more powerful tool in future work. My own practice is to found this course on Chrystal's *Algebra* and make numerous references to other works. It should begin at the beginning of the subject, and give a solid foundation in the fundamental laws and processes of algebra, perhaps the most difficult part of the course for the average student. A topical outline of the course is hardly necessary here, as it will be varied somewhat to meet the needs of different classes. It may well be extended to include the most essential parts of theory of equations, and the general theory of integral functions. As the object is to give facility of execution, quite as much as to teach the theory of the subject, a considerable list of select problems should be assigned, and carefully solved.

For the next three hours I would choose the subject geometry.

Geometry. This course should probably begin with a short review of the elementary course in analytic geometry, treated in a more scientific manner. From this we proceed to homogeneous point and line coördinates and the principle of duality. Here a short account of the general treatment given in courses in geometry of position, where point, line and plane, as fundamental elements lead to a multiplicity of dualistic theorems, will widen the student's horizon, and possibly awaken a desire for a later course in projective geometry. Dropping back to the point and line, the theorems on the complete quadrilateral and harmonic division are followed by the more general theorems on the descriptive properties of conics by Pascal, Chasles and others. The study of the metric properties brings in the line at infinity and the circular points at infinity. A study of projection, cross ratio and involution will complete a course sufficiently difficult for all but the best grounded classes in the subject.

For the next three hours, that is, the first term of the senior year, I would have a course in advanced calculus.

Advanced Calculus. This should begin with a brief review of the elementary course, and take up the subjects in differential calculus that were omitted or taken too briefly in the first course. Here partial differentiation should receive careful treatment. The work in infinite series, Taylor's theorem, etc., given in the first course may be considerably extended. But at least two thirds of the time should be devoted to the integral calculus. The definite integral may here be treated in a more advanced way than was possible in the first course. The student should now receive careful, though necessarily brief instruction in such

subjects as the gamma functions, elliptic integrals, line, surface and space integrals, and his knowledge of the applications of the definite integral should be considerably extended. It has been my custom to found the integral part of this course on Volume 2 of Byerly's *Calculus*, and to make liberal references to other works for both theory and application.

For the last three hours I find it difficult to decide on a subject. There are many subjects entirely suitable that would add variety to the somewhat restricted schedule given above. In many ways a course in analytic mechanics would be ideal at this time. However, keeping the graduate school in mind as the goal toward which the student is working, and recalling Dr. Bolza's rather insistent claim that a student should have two years of calculus for admission to graduate work, I venture to suggest a course in differential equations.

Differential Equations. This can be made as difficult as the ability of the class will permit, and furnishes a splendid review in the processes of integration. No outline of this need be suggested, for it will of necessity be kept quite elementary. However it should be given as many points of contact with other subjects as time will permit.

This schedule, three hours of algebra, three hours of geometry, and six hours of calculus, will probably seem to lack variety. I have kept the graduate school in mind and to the best of my ability selected subjects accordingly. For those students whose school career must stop with the bachelor's degree, or who pursue other lines of study later, a different selection may be advisable.

Conclusion. While it is not directly pertinent to the question, I venture to suggest that some students enter college prepared to begin analytic geometry, and are thus in line for eighteen hours instead of twelve. This is by no means uncommon in the college with which I am connected, owing to the good work of the mathematics department of the Topeka high school. Others will frequently desire to carry two courses simultaneously. Hence there is considerable opportunity to introduce variety where the teaching force will permit. Also a number of two hour courses can be offered, considerably widening the scope of the above schedule. In this way such subjects as history of mathematics, theory of determinants, geometry of position, geometry of three dimensions, and many others may be introduced, and the student given a much wider view without subtracting from his foundation work.

NOTES AND NEWS.

The problem department of SCHOOL SCIENCE AND MATHEMATICS is now in charge of DR. J. O. HASSLER, of the Englewood High School, Chicago.

MR. CARL GARLOUGH has been appointed to an instructorship in mathematics in Wheaton College, Wheaton, Ill.

PROFESSOR FLORIAN CAJORI has an interesting article in *Bibliotheca Mathematica* for May, 1915, on "An integration antedating the integral calculus."

DR. U. G. MITCHELL gave an address recently before the teachers of Marion county, Kansas, on "A new point of view for text-books and teachers."

The American Mathematical Society met in regular session at Columbia University, New York City, on October thirtieth; the southwestern section met at Washington University, St. Louis, on November twenty-seventh.

"Number and the quadratic" is the title of a paper by PROFESSOR RICHARD MORRIS, of Rutgers College, appearing in the November number of *School Science and Mathematics.*

"The need of supervision in college teaching" is the title of an article in *School and Society* for October 9, 1915, which is worthy of consideration by all college teachers, not least by teachers of mathematics.

The University of Washington Publications in mathematical and physical sciences have been inaugurated with an extensive presentation of "An arithmetical theory of certain numerical functions" by ERIC T. BELL.

In *Science* for July 23 PROFESSOR W. H. ROEVER gives a comparative and critical *résumé* of the treatments of the meridional deviation of a falling body by himself, by PROFESSOR F. R. Moulton, and by DR. R. S. Woodward, already noted in early numbers of the MONTHLY.

An inquiry made several years ago in *L'Intermédiaire* by MR. E. B. ESCOTT for a complete study of the equation $aX^4 + bY^4 = cZ^2$ brought forth a very extended article by A. GÉRARDIN in the July number of that journal, in which he solves the problem by three different methods and supplements this by a bibliography.

PROFESSOR G. A. MILLER writes on "A few classic unknowns in mathematics" in the October number of *The Scientific Monthly.* In this article he lists several well known questions in the theory of numbers which are as yet unsolved but the nature of which can be understood even by persons quite limited in their knowledge of mathematics.

FRANKLIN, MACNUTT and CHARLES have published a "supplement" which is to take the place of the first forty-one pages of the 1913 edition of their text in calculus. The principal change consists in giving a more rigorous introduction for the derivative and differential notation. A list of corrections to the earlier edition is also given.

DR. G. J. KOLLEN, president emeritus of Hope College, Holland, Michigan,

died on September 5. For fifty years he had been in educational work, being at various times professor of mathematics, natural philosophy and applied mathematics. He was a graduate of Hope College, and held the honorary degree of doctor of laws from Rutgers College.

"The Human Significance of Mathematics" is the title of an address given by PROFESSOR C. J. KEYSER at Berkeley, Calif., August 3, 1915, at a joint meeting of the American Mathematical Society, the American Astronomical Society, and Section A of the American Association for the Advancement of Science. The address is printed in *Science* for November 22, 1915. It is well worth the attention of every teacher of mathematics.

The Macmillan Company announces the publication of a manual of "Constructive Geometry" by PROFESSOR E. R. HEDRICK, a "Grammar School Arithmetic" by PROFESSOR F. CAJORI, and an "Elementary Algebra" by PROFESSOR CAJORI and LETITIA R. ODELL, the latter of the North Side High School of Denver.

In the October number of the *School Review* MR. RALEIGH SCHORLING, of the University of Chicago High School, reports the results of a survey made for the purpose of investigating the general problem of individual differences in the teaching of secondary school mathematics, as treated in the various institutions which offer practice-teaching courses. The article concerns more particularly the methods for dealing with the slower students.

The March–April number of *Rendiconti del Circolo Matematico di Palermo* contains an unusually large proportion of articles by Americans. These are "Aggregates of minors of persymmetric determinants" by PROFESSOR W. H. METZLER and L. H. RICE, "A certain class of transcendental curves" by DR. FLORENCE E. ALLEN, "Conjugate systems with equal tangential invariants and the transformation of Moutard" by PROFESSOR L. P. EISENHART, and a "Note on trigonometric interpolation" by DR. DUNHAM JACKSON.

Upon the invitation of the American Society of Mechanical Engineers, a joint committee representing seventeen scientific societies, including the American Mathematical Society, has been working on the question of "Standards for graphic representation." A brief preliminary report has been published for the purpose of soliciting criticisms and suggestions. Copies may be obtained from the American Society of Mechanical Engineers, 29 West 39th St., New York City. The price is ten cents. The representative of the American Mathematical Society is PROFESSOR H. E. HAWKES, of Columbia University.

The Carnegie Endowment for International Peace has just issued a pamphlet entitled "Problems about war for classes in arithmetic." This was prepared by Professor DAVID EUGENE SMITH, and will be sent to any teacher who requests a

copy. Letters should be sent to the Endowment, 407 West 117th Street, New York City. The purpose of the pamphlet is to bring before pupils at an impressionable age some idea of their responsibility for a wiser use of money on the part of the whole world than that which looks for a model to the present vast expenditures for armaments.

At Rutgers College a Mathematical Club has been formed among the undergraduates, and during the past year the following papers were presented:

"The cubic and the quartic," MR. EDWIN FLORANCE; "The fundamental theorem of algebra," MR. LOUIS B. GITTLEMAN; "Elementary theorems on the roots of an equation," MR. J. H. HUNTINGTON; "The fourth dimension," MR. JAMES B. SCARR; "The elements of life insurance," PROFESSOR STANLEY E. BRASEFIELD; "Arithmetical processes and the number field," DR. RICHARD MORRIS; "The solution of certain trigonometric equations," MR. HAROLD I. FAWCETT; "Some trigonometric series," MR. CLIFFORD P. OSBORNE; "Trilinear coördinates," DR. JOHN A. INGHAM. The MONTHLY comes to the College library and is read by the mathematical students and others.

The fifty-third annual meeting of the Minnesota Educational Association was held at Minneapolis, October 27–30, 1915. The mathematics section of the secondary department held two sessions, at one of which MR. W. D. REEVE, the newly appointed head of the department in the University of Minnesota high school, discussed a list of helpful books and journals for teachers of mathematics. Also, at the section for college teachers of education MR. REEVE gave a paper on a methods course for training high-school teachers of mathematics. At each of these sessions, PROFESSOR H. E. SLAUGHT, being called from the floor, laid emphasis upon individual stimulus for high-school teachers above their daily routine and upon the need of some systematic attention to the preparation of college instructors in mathematics with respect to the principles of good teaching.

The *Mathematics Teacher* for September, 1915, contains an extensive report of a committee on bibliography of which EUGENE R. SMITH is chairman. The list of books given covers many things which might be of interest to a teacher of secondary mathematics and which cannot be found in the "Bibliography of the teaching of mathematics" published by the U. S. Bureau of Education as Bulletin No. 503. In particular, it contains lists of text-books and exercise books in algebra, geometry, and trigonometry; publications of state departments, and of the U. S. Bureau of Education, relating to mathematics in the secondary schools; and association reports and college publications on the teaching of algebra, geometry, and trigonometry; covering in all twenty-two closely printed pages. This bibliography should be of great service to all teachers of mathematics.

According to the annual report in *Science* for October 22, 1915, on Doctorates in American Universities, there were 23 doctorates in mathematics conferred

during 1915. The average number for the past ten years is twelve per year, and the total to date is 297. The departments conferring more than ten degrees in 1915 are, in order, chemistry 85, botany 40, zoology 32, physics 31, geology 26, mathematics 23 and psychology 22. The universities conferring two or more degrees in mathematics in 1915 are, in order, Chicago 7, Harvard 3, Columbia, Cornell, Pennsylvania, and Yale each 2. Illinois, Johns Hopkins, Missouri, and Princeton each conferred one doctorate in mathematics. The total number of doctorates in the sciences for 1915 was 309 and the grand total in all branches was 556. The number conferred during the past eighteen years is 6,320.

It is altogether in line with the purposes of the MONTHLY to call attention to notable texts in higher mathematics as they appear. In this connection we may mention among the earlier numbers of the Edinburgh Mathematical Tracts, published by G. Bell and Sons, under the general editorship of PROFESSOR E. T. WHITTAKER, the following numbers: "A course in interpolation and numerical integration for the mathematical laboratory" by DAVID GIBB; "Relativity" by A. W. CONWAY; "A course in Fourier's analysis and periodogram analysis for the mathematical laboratory" by G. A. CARSE and G. SHEARER; "A course in the solution of spherical triangles for the mathematical laboratory" by H. BELL; and "An introduction to the theory of automorphic functions" by L. R. FORD. The majority of the Edinburgh Tracts were inspired by the work of the mathematical laboratory in Edinburgh and are devoted to laboratory instruction, in the belief that "the student trained in modern pure mathematics is somewhat helpless in face of the concrete problems presented by the applied mathematical sciences" unless his training is supplemented by courses in "a mathematical laboratory, devoted to numerical and graphical calculation and analysis."

The fifteenth annual meeting of the Central Association of Science and Mathematics Teachers was held in Chicago on Friday and Saturday, November 26, 27. At the opening of the general session, Professor E. R. HEDRICK, of the University of Missouri, gave an address on "Required mathematics in secondary schools and colleges"; and EDWARD A. STEINER, professor of sociology at Grinnell College, Iowa, spoke on "A message from a sociologist to teachers of science and mathematics."

The mathematics section held two sessions at which committee reports were presented: on algebra by MARY JACKSON of Lincoln High School, Nebraska; on geometry by E. R. BRESLICH of the University of Chicago High School; on correlation of secondary mathematics by EDNA ALLEN of Iowa State Teachers College; and on vocational mathematics by KENNETH G. SMITH of Iowa State Agricultural College. There were also two addresses at the section meetings; one by Professor G. W. MYERS of the University of Chicago on "Current educational movements and general mathematics," and the other by Professor L. C. KARPINSKI of the University of Michigan on "The history of algebra, with stereopticon views." There was also a round table discussion on "History of

mathematics—its place and value in secondary mathematics," led by CHRISTINE BEDNAR, of Parker High School, Chicago, and followed by Professor KARPINSKI and others.

The call for an organization meeting of a new National Mathematical Association was signed by about four hundred and fifty persons representing every state in the Union, the District of Columbia, and Canada, and including high school, normal school, college, and university teachers, consulting engineers, actuaries, and others who are interested in mathematics purely for its own sake. The meeting was scheduled for Thursday, December 30, at ten o'clock A. M., in Page Hall, Ohio State University. Doubtless other sessions will be needed to complete the organization. Only one program was arranged, aside from the business sessions; namely, on Friday morning at nine o'clock, Professor L. C. Karpinski, of the University of Michigan, is to give an illustrated address on "The Story of Algebra," to which all mathematicians in attendance upon the Columbus meetings are invited. A full report of the organization of the new Association will be printed in the January issue of the MONTHLY, and a copy will be sent to all persons who signed the call and to all who request the same before the date of the meeting. Announcement of conditions of membership will be sent to all who have manifested interest in this movement, and the announcement stated that a proposition would be made at Columbus to admit to charter membership all who may join the new Association before April 1, 1916.

The MONTHLY closes its third year since its reorganization with many of its earlier fears and anxieties removed, with its subscription list more than trebled, with a large and increasing body of contributors, and with a feeling of certainty that no mistake was made in regard to the opportunity that seemed to be offered in the field in which its operations have been centered. High praise is due to those colleges and universities which deemed the cause represented by the MONTHLY to be worthy of their consideration and which made possible the promotion of this cause by their generous subsidy contributions throughout the three-year period. Hearty thanks are due the representatives of these institutions who have constituted the Board of Editors and who have unselfishly given their time and their services, amounting in the aggregate to no small consideration, thus showing their faith in this cause and their willingness to back it by works. Full credit is due the constituency of the MONTHLY, a widely representative body of persons who have shown their confidence and indicated their loyalty by their continued adherence in ever-increasing numbers. Finally, and not least, credit is due those who have made their contributions to the cause by purchasing advertising space, and to The New Era Printing Company, whose excellent composition and presswork are worthy of mention.

We believe that we can confidently promise for 1916 the continuation of all the worthy features in the make-up of the MONTHLY, the elimination of many of its faults, and the addition of several things which will increase its interest and broaden its influence.